INTRODUCTION TO THE BOUNDARY CONTROL METHOD

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1. Introduction

The Boundary Control method can be used to solve several coefficient determination problems for wave equations. The method originates from [5], and it allows for building a rather comprehensive theory of inverse problems of coefficient determination type. Indeed, many such problems can be reduced to inverse boundary spectral problems, which again can be reduced to coefficient determination problems for wave equations. This is the case for inverse problems for heat and non-stationary Schroödinger equations [6], as well as for several time-fractional [7] and space-fractional [8] equations. Inverse problems for linear elliptic equations on a wave guide [9], and some (very special) non-linear elliptic equations [10] are also covered by this theory.

The method has been generalized for symmetric systems of wave equations with scalar leading order [11] and for wave equations on Lorentzian manifolds satisfying a curvature bound [12].

The purpose of these lecture notes is to give an introductory exposition of the method. For an in-depth presentation, we refer to the monograph [3].

2. A model problem in $1+1$ dimensions

Consider the following initial-boundary value problem

$$\begin{cases} (\partial_t^2 - \partial_x^2 + q)u = 0; \quad \text{on } \mathbb{R}^+ \times (0,1); \\ u|_{x=0} = f; \\ u|_{x=1} = 0; \\ u|_{t=0} = \partial_t u|_{t=0} = 0, \end{cases} \tag{1}$$

where f is a boundary source, q is a potential, and $\mathbb{R}^+$ stands for the interval $(0,\infty)$. We denote its solution by $u^f = u(t,x)$. Define the Dirichlet-to-Neumann map

$$\Lambda f = \partial_x u^f\big|_{x=0}. \tag{2}$$

Then the inverse problem is stated as follows: recover the potential q, given the operator Λ.

2.1. Finite speed of propagation in the case $q = 0$.

The finite speed of propagation property for the acoustic wave equation plays a central role in both direct and inverse theory. To illustrate the property in the simplest possible case, let us consider (1) with $q = 0$, that is,

$$(\partial_t^2 - \partial_x^2)u = 0. \tag{3}$$

We set

$$\mathcal{E}(t,x) = |\partial_t u(t,x)|^2 + |\partial_x u(t,x)|^2.$$

Theorem 1 (Conservation of energy). *Suppose that* $u \in C^2(\overline{\mathbb{R}^+ \times (0,1)})$ *satisfies*

$$\begin{cases} (\partial_t^2 - \partial_x^2)u = 0 \quad \text{on } \mathbb{R}^+ \times (0,1) \\ u\,|_{x=0} = u\,|_{x=1} = 0. \end{cases}$$

Then the global energy

$$E(t) = \frac{1}{2}\int_0^1 \mathcal{E}(t,x)dx$$

satisfies $\partial_t E(t) = 0$ *for all* $t > 0$.

Proof. We write,

$$\partial_t E(t) = \frac{1}{2}\int_0^1 \partial_t \mathcal{E}(t,x)dx$$
$$= \int_0^1 \left(\partial_t u(t,x)\partial_t^2 u(t,x) + \partial_x u(t,x)\partial_t\partial_x u(t,x)\right) dx.$$

By integration by parts, we obtain

$$\partial_t E(t) = [\partial_x u(t,x)\partial_t u(t,x)]_0^1$$
$$+ \int_0^1 \left(\partial_t^2 u(t,x) - \partial_x^2 u(t,x)\right)\partial_t u(t,x)dx.$$

The first term is zero since $u(t,0) = u(t,1) = 1$ and the last integral is zero since u satisfies the wave equation. □

Theorem 2 (Finite speed of propagation)**.** *Suppose that* $u \in C^2(\overline{\mathbb{R}^+ \times (0,1)})$ *satisfies*

$$\begin{cases} (\partial_t^2 - \partial_x^2)u = 0 & \text{on } \mathbb{R}^+ \times (0,1) \\ u \mid_{x=1} = 0. \end{cases}$$

Then the local energy

$$E(t) = \frac{1}{2}\int_t^1 \mathcal{E}(t,x)dx$$

satisfies $\partial_t E(t) \le 0$ *for all* $t > 0$.

Proof. We write, by using Leibniz integral rule,

$$\partial_t E(t) = -\frac{1}{2}\mathcal{E}(t,t) + \frac{1}{2}\int_t^1 \partial_t \mathcal{E}(t,x)dx$$
$$= -\frac{1}{2}\mathcal{E}(t,t) + \int_t^1 \left(\partial_t u(t,x)\partial_t^2 u(t,x) + \partial_x u(t,x)\partial_t\partial_x u(t,x)\right) dx.$$

An integration by parts gives

$$\partial_t E(t) = -\frac{1}{2}\mathcal{E}(t,t) + [\partial_x u(t,x)\partial_t u(t,x)]_t^1$$
$$+ \int_t^1 \left(\partial_t^2 u(t,x) - \partial_x^2 u(t,x)\right)\partial_t u(t,x)dx.$$

Since u is the solution of the wave equation the last integral is 0. Moreover, since $u(t,1)=0$, the second term is 0 at 1, so that

$$\partial_t E(t) = -\frac{1}{2}\mathcal{E}(t,t) - \partial_t u(t,t)\partial_x u(t,t) = -\frac{1}{2}\left(\partial_t u(t,t) + \partial_x u(t,t)\right)^2 \leq 0.$$

□

In particular, if $E(0)=0$ then $E(t)=0$ for all $t>0$. So if u and $\partial_t u$ vanish initially on the right half line $\{x>0\}$ then at time t they vanish on the half line $\{x>t\}$. To see that this statement is optimal, observe that $u(t,x)=f(x-t)$ is a solution to (3) for any $f\in C^2(\mathbb{R})$, and choose $f(r)$ vanishing for $r>0$ but not near $r=0$.

2.2. **Finite speed of propagation for general q.** Let us now consider a wave equation with a potential as in (1), and prove a finite speed of propagation result in the very natural setting of a diamond, see the set K in Theorem 3 below. It is also possible to study the case where boundary conditions are posed, say at $x=0$ and $x=1$ as before, and where the diamond intersects the boundary of $\mathbb{R}^+\times(0,1)$. However, we leave this case to the reader.

Theorem 3 (Finite speed of propagation)**.** *Let $X>0$ and define*

$$K := \{(t,x)\in\mathbb{R}\times\mathbb{R} : |x| \leq X - |t|\}.$$

Let $q\in L^\infty(K)$ and let $u\in C^2(K)$ be a solution of the equation

$$\begin{cases} \left(\partial_t^2 - \partial_x^2 + q(t,x)\right)u = 0, & \text{on } K;\\ u\,|_{t=0} = \partial_t u\,|_{t=0} = 0, & \text{on } (-X,X). \end{cases}$$

Then $u|_K = 0$.

Proof. We will consider the case $t>0$. The case $t<0$ is analogous and we omit its proof. Let us set $I(t)=[-X+t, X-t]$ and define the energy

$$E(t) := \frac{1}{2}\int_{I(t)} \mathcal{E}(t,x)dx.$$

Using Leibniz integral rule, we obtain

$$\begin{aligned}\partial_t E(t) = &-\frac{1}{2}\left(|\partial_t u(t,X-t)|^2 + |\partial_x u(t,X-t)|^2\right)\\ &-\frac{1}{2}\left(|\partial_t u(t,-(X-t))|^2 + |\partial_x u(t,-(X-t))|^2\right)\\ &+\int_{I(t)}\left(\partial_t u(t,x)\partial_t^2 u(t,x) + \partial_x u(t,x)\partial_t\partial_x u(t,x)\right)dx.\end{aligned}$$

An integration by parts gives

$$
\begin{aligned}
\partial_t E(t) = &-\frac{1}{2}\left(|\partial_t u(t,X-t)|^2 + |\partial_x u(t,X-t)|^2\right)\\
&-\frac{1}{2}\left(|\partial_t u(t,-(X-t))|^2 + |\partial_x u(t,-(X-t))|^2\right)\\
&+\partial_t u(t,X-t)\partial_x u(t,X-t) - \partial_t u(t,-(X-t))\partial_x u(t,-(X-t))\\
&+\int_{I(t)} \partial_t u(t,x)\left(\partial_t^2 u(t,x) - \partial_x^2 u(t,x)\right)dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\partial_t E(t) = &-\frac{1}{2}\Big(|\partial_t u(t,X-t)| - |\partial_x u(t,X-t)|\Big)^2\\
&-\frac{1}{2}\Big(|\partial_t u(t,-(X-t))| + |\partial_x u(t,-(X-t))|\Big)^2\\
&+\int_{I(t)} \partial_t u(t,x)\left(\partial_t^2 u(t,x) - \partial_x^2 u(t,x)\right)dx.
\end{aligned}
$$

Since the first two terms are non-positive, it follows

$$
\begin{aligned}
\partial_t E(t) \le &-\int_{I(t)} \partial_t u(t,x) q(t,x) u(t,x) dx\\
&+\int_{I(t)} \partial_t u(t,x)\left(\partial_t^2 u(t,x) - \partial_x^2 u(t,x) + q(t,x)u(t,x)\right)dx.
\end{aligned}
$$

Let us use notation

$$P := \partial_t^2 - \partial_x^2 + q.$$

The Cauchy-Schwarz inequality imply

$$
\begin{aligned}
&E(t) = E(0) + \int_0^t \partial_t E(s) ds\\
&\le E(0) + 2\int_0^t \int_{I(s)} \left(|\partial_t u(s,x)|^2 + |q(s,x)u(s,x)|^2 + |Pu(s,x)|^2\right) dxds.
\end{aligned}
$$

Hence,

$$E(t) \le E(0) + C\left(\int_0^t E(s)ds + \int_0^t \int_{I(s)} \left(|u(s,x)|^2 + |Pu(s,x)|^2\right) dxds\right), \tag{4}$$

where C is a constant depending on q and X, which will change from line to line.

Let us now take care of the $|u|^2$ term caused by the potential q. We will use arguments close to the proof in [2, Sec. 4.2]. For $x \in \mathbb{R}$ fixed, we know

$$u(t,x) = u(0,x) + \int_0^t \partial_t u(s,x) ds.$$

Let us square it, and then, use Cauchy's inequality and the Cauchy-Schwarz inequality to obtain

$$|u(t,x)|^2 \leq 2|u(0,x)|^2 + 2t \int_0^t |\partial_t u(s,x)|^2 ds. \tag{5}$$

Next, we define

$$z(t) := E(t) + \int_{I(t)} |u(t,x)|^2 dx.$$

Using (4) and (5), we estimate

$$z(t) \leq E(0) + C\left(\int_0^t E(s)ds + \int_0^t \int_{I(s)} \left(|u(s,x)|^2 + |Pu(s,x)|^2\right) dxds\right)$$
$$+ 2\int_{I(t)} |u(0,x)|^2 dx + 2t \int_{I(t)} \int_0^t |\partial_t u(s,x)|^2 dsdx.$$

Since we consider $t \in [0, X]$, and since $I(t) \subset I(s)$ for $s \in [0,t]$, the last term can be estimated from above by $2X \int_0^t E(s)ds$. Therefore, we have

$$z(t) \leq Cz(0) + C\int_0^t z(s)ds + C\int_0^t \int_{I(s)} |Pu(s,x)|^2 dxds.$$

Using the Gronwall's inequality in the integral form[1], we obtain

$$z(t) \leq C\left(z(0) + \int_0^t \int_{I(s)} |Pu(s,x)|^2 dxds\right). \tag{6}$$

Since $z(0) = 0$ and $Pu = 0$, we conclude that $z(t) = 0$ for $t \in [0, X]$. Recalling the definition of the function z, we see that $u(t,x) = 0$ for $t \in [0, X]$ and $x \in I(t)$. □

Theorem 4 (Unique continuation)**.** *Let $T > 0$ and define*

$$K := \{(t,x) \in [-T,T] \times \mathbb{R} : |x| \leq T - |t|\}.$$

Let $q \in L^\infty(K)$ and let $u \in C^2(K)$ be a solution of the equation

$$\begin{cases} \left(\partial_t^2 - \partial_x^2 + q(t,x)\right) u = 0, & \text{on } \mathbb{R} \times \mathbb{R}; \\ u \mid_{x=0} = \partial_x u \mid_{x=0} = 0, & \text{on } [-T,T]. \end{cases}$$

[1] [1, Appendix B.2.k]

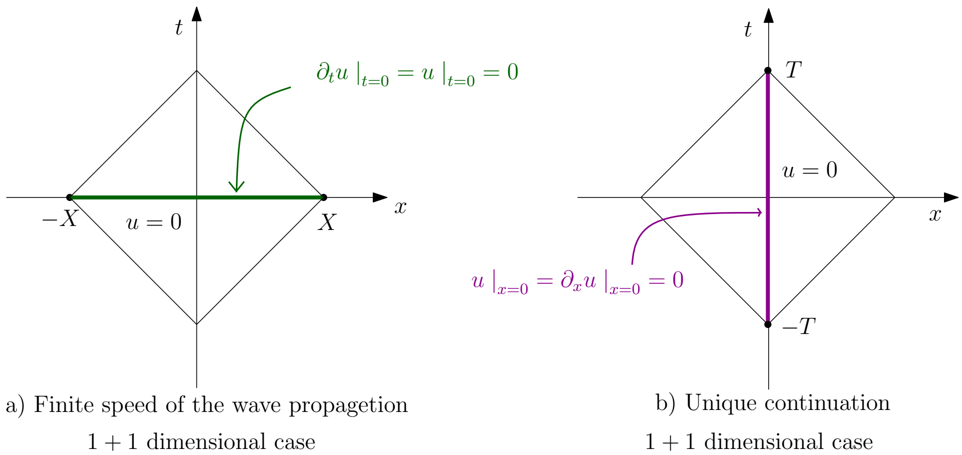

FIGURE 1.

Then $u|_K = 0$.

Proof. By interchanging the roles of t and x, we see that the theorem coincides with Theorem 3; see Figure 1. □

2.3. **Homework: direct problem for the wave equation.** We consider the equation

$$(7) \qquad \begin{aligned} &(\partial_t^2 - \Delta + q(x))u(t,x) = f(t,x), \quad \text{in } (0,T)\times\mathbb{R}^n, \\ &u|_{t=0} = \partial_t u|_{t=0} = 0, \end{aligned}$$

where $q \in C_0^\infty(\mathbb{R}^n)$. The proof of finite speed of propagation generalizes to higher dimensions (but the proof of unique continuation does not).

Theorem 5 (Finite speed of propagation, nD)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be an open set. Assume that* $u \in C^2(\mathbb{R}\times\mathbb{R}^n)$ *is a solution of the equation*

$$\begin{cases} (\partial_t^2 - \Delta + q(x))\,u = 0, & \text{on } \mathbb{R}\times\mathbb{R}^n; \\ u\,|_{t=0} = \partial_t u\,|_{t=0} = 0, & \text{on } \Omega. \end{cases}$$

Then

$$u|_{\mathcal{C}} = 0,$$

where

$$\mathcal{C} := \{(t,x) \in \mathbb{R}\times\mathbb{R}^n : d(x, \mathbb{R}\setminus\Omega) > |t|\}.$$

Homework 1. *Write* $\Box = \partial_t^2 - \Delta$,

$$I(s) = \{(t,x) \in \mathcal{C} : t = s\},$$

and define z *analogously to the 1D case. Show that*

$$z(t) \le C \left(z(0) + \int_0^t \int_{I(s)} |(\Box + q)u(s,x)|^2 dx ds \right). \tag{8}$$

Hint: The case $q = 0$ *is proven in* [1].

Homework 2. *Prove Theorem 5.*

2.3.1. *Existence of solutions.* Let $f \in C_0^\infty((0,T) \times \mathbb{R}^n)$. Then there are $R, \epsilon > 0$ such that

$$\operatorname{supp}(f) \subset (\epsilon, T) \times B(0,R).$$

We use the shorthand notation

$$B = B(0,R), \quad K = \bigcup_{t \in (\epsilon, T)} B(0, R+t).$$

Choose $\chi \in C_0^\infty((0,\infty) \times \mathbb{R}^n)$ such that $\chi = 1$ near K, and define the spaces

$$X_0 = C_0^\infty((-\infty,T) \times \mathbb{R}^n), \quad Y_0 = \{\chi(\Box + q)w;\ w \in X_0\}.$$

Homework 3. *By considering the function* $u(t) = w(T-t)$*, verify that the finite speed of propagation estimate (8) implies*

$$\|w(t)\|_{H^1(B)} + \|\partial_t w(t)\|_{L^2(B)} \le C\|(\Box + q)w\|_{L^2(K)}, \quad t \in (\epsilon, T),\ w \in X_0.$$

In particular,

$$\|w\|_{L^2(\operatorname{supp}(f))} \le C\|\chi(\Box + q)w\|_{L^2(\mathbb{R}^{1+n})}, \quad w \in X_0.$$

We define a linear map $L_0 : Y_0 \to \mathbb{R}$ by

$$L_0(\chi(\Box + q)w) = \langle f, w \rangle.$$

Here $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^{1+n})}$ is the inner product in $L^2(\mathbb{R}^{1+n})$, and we write $\|\cdot\|$ for the associated norm. Note that L_0 is well-defined, since

$$|L_0(\chi(\Box + q)w)| \le \|f\|\|w\|_{L^2(\operatorname{supp}(f))} \le C\|f\|\|\chi(\Box + q)w\|.$$

The Hahn-Banach theorem[2] implies that there is a continuous linear functional L on $L^2(\mathbb{R}^{1+n})$ that coincides with L_0 on Y_0. The Riesz representation theorem[3] implies that there is $v \in L^2(\mathbb{R}^{1+n})$ such that

$$L(\phi) = \langle v, \phi \rangle, \quad \phi \in L^2(\mathbb{R}^{1+n}).$$

[2] `https://en.wikipedia.org/wiki/Hahn%E2%80%93Banach_theorem`

[3] `https://en.wikipedia.org/wiki/Riesz_representation_theorem`

Thus for $w \in X_0$,

$$\langle v, \chi(\Box + q)w\rangle = L(\chi(\Box + q)w) = L_0(\chi(\Box + q)w) = \langle f, w\rangle,$$

We define $u = \chi v$. Then $u = 0$ near $t = 0$.

Homework 4. *Verify that $(\Box + q)u = f$ in the sense of distributions on $(0, T) \times \mathbb{R}^{1+n}$.*

Note that we can not right away use (8) to conclude that u is the unique solution of (7), since at this point we have shown (8) only for functions in $C^2(\mathbb{R}^{1+n})$. For this reason we need to show that u has better regularity properties than just L^2.

2.3.2. *Regularity.* A very highbrow way to show that the solution u obtained above is actually smooth is to use propagation of singularities for the wave equation. We will outline a more elementary proof though.

Define

$$X = \{v \in C^\infty((0,T)\times\mathbb{R}^{1+n});\ \mathrm{supp}(v) \subset (0,T]\times\mathbb{R}^n,\ \mathrm{supp}((\Box+q)v) \subset (0,T]\times B\}.$$

Homework 5. *Show that (8) implies for all $v \in X$ that* $\mathrm{supp}(v) \subset (0, T] \times \overline{B(0, R + T)}$ *and that*

$$\|v\|_{C(0,T;H^1(\mathbb{R}^n))} + \|v\|_{C^1(0,T;L^2(\mathbb{R}^n))} \le C\|(\Box + q)v\|_{L^2((0,T)\times B)}. \tag{9}$$

Homework 6. *By using the Plancherel theorem*[4] *show the following lemma:*

Lemma 1. *Let $\phi \in L^2(\mathbb{R}^n)$ and suppose that $\Delta\phi \in L^2(\mathbb{R}^n)$. Then $\phi \in H^2(\mathbb{R}^n)$ and*

$$\|\phi\|_{H^2(\mathbb{R}^n)} \le C\|(1 - \Delta)\phi\|_{L^2(\mathbb{R}^n)}.$$

Let $f \in C_0^\infty((0,T) \times \mathbb{R}^n)$ and let $u \in L^2(\mathbb{R}^{1+n})$ satisfy (7) in the sense that $(\Box + q)u = f$ as distributions and $u = 0$ near $t = 0$. By the discussion in the previous section, such u exists.

Set $u_j = \eta_{1/j} * u$, that is,

$$u_j(t) = \int_{\mathbb{R}} \eta_{1/j}(t - s)u(s)ds, \quad j = 1, 2, \dots,$$

where η_ϵ is the mollifier as in [1, Appendix C.5]. Then $u_j \in C^\infty(\mathbb{R}; L^2(\mathbb{R}^n))$ and $u_j = 0$ near $t = 0$ for large j. Define also $f_j = \eta_{1/j} * f$.

Homework 7. *Show that $(\Box + q)u_j = f_j$.*

Then

$$\Delta u_j(t) = \partial_t^2 u_j(t) + qu_j(t) - f_j(t) \in L^2(\mathbb{R}^n),$$

and therefore Lemma 1 implies that $u_j \in C^\infty(\mathbb{R}; H^2(\mathbb{R}^n))$.

[4] `https://en.wikipedia.org/wiki/Plancherel_theorem`

Homework 8. *Use an induction to show that* $u_j \in C^\infty(\mathbb{R}; H^{2k}(\mathbb{R}^n))$ *for all* $k = 1, 2, \dots$.

Then the Sobolev embedding theorem[5] implies that $u_j \in C^\infty(\mathbb{R}^{1+n})$. In particular, $u_j \in X$ for large j.

Homework 9. *Apply (9) to the difference* $u_j - u_k$ *and show that* $(u_j)_{j=1}^\infty$ *is a Cauchy sequence in the Banach space*

$$\mathcal{H}^1 = C(0,T; H^1(\mathbb{R}^n)) \cap C^1(0,T; L^2(\mathbb{R}^n)). \tag{10}$$

Thus it converges, say to u_∞, in this space. But $u_j \to u$ in the larger space $L^2(0,T; L^2(\mathbb{R}^n))$, and therefore $u = u_\infty$. In particular u is in the space $\mathcal{H}^1$.

Homework 10. *Apply (9) to* u_j *and let* $j \to \infty$ *to show that (9) holds with* $v = u$. *Conclude that (7) has a unique solution in* $\mathcal{H}^1$.

Define the linear map

$$W_0 : C_0^\infty((0,T) \times B) \to \mathcal{H}^1, \quad W_0 f = u,$$

where u is the solution of (7).

Homework 11. *Show that (9) implies that* W_0 *has a unique continuous extension*

$$W : L^2((0,T) \times B) \to \mathcal{H}^1. \tag{11}$$

2.3.3. *Higher regularity.* Without higher regularity results we are forced to work with solutions in the sense of distributions, and this can be cumbersome.

Homework 12. *Show that* $v \in X$ *implies* $\partial_t^k v \in X$ *for all* $k = 1, 2, \dots$. *Use this and Lemma 1 to show*

$$\sum_{k=0}^{m+1} \|v\|_{C^k(0,T; H^{m+1-k}(\mathbb{R}^n))} \le C \sum_{k=0}^{m} \|\partial_t^k (\Box + q) v\|_{L^2(0,T; H^{m-k}(\mathbb{R}^n))}, \quad v \in X. \tag{12}$$

Homework 13. *Show that* $W_0 : C_0^\infty((0,T) \times B) \to C^\infty((0,T) \times \mathbb{R}^n)$. *A similar proof can be found in* [1, Th. 7.2.3.6].

2.4. **An integration by parts trick for the inverse problem.** We begin by recalling that u^f and u^h are solutions of (1), when the boundary source are given by f and g, respectively. Let us define the function

$$W_{f,h}(t,s) := (u^f(t,\cdot), u^h(s,\cdot))_{L^2(0,1)}.$$

Then the following holds

[5] https://en.wikipedia.org/wiki/Sobolev_inequality#Sobolev_embedding_theorem

Lemma 2. *Let f, $h \in C_0^\infty(\mathbb{R}^+)$. The operator Λ determines $W_{f,h}$.*

Proof. Since u^f, and u^h are solutions of (1), it follows

$$(\partial_t^2 - \partial_s^2) W_{f,h}(t,s) = \left(\partial_x^2 u^f(t,\cdot) - q u^f(t,\cdot), u^h(s,\cdot)\right)_{L^2(0,1)} - \left(u^f(t,\cdot), \partial_x^2 u^h(s,\cdot) - q u^h(s,\cdot)\right)_{L^2(0,1)}.$$

Further, using the integration by parts, we obtain

$$(\partial_t^2 - \partial_s^2) W_{f,h}(t,s) = \partial_x u^f(t,1) u^h(s,1) - \partial_x u^f(t,0) u^h(s,0) - u^f(t,1)\partial_x u^h(s,1) + u^f(t,0)\partial_x u^h(s,0).$$

Boundary conditions for u^f and u^h give

$$(\partial_t^2 - \partial_s^2) W_{f,h}(t,s) = f(t)\Lambda h(s) - \Lambda f(t) h(s). \tag{13}$$

Let us denote the right-hand side of (13) by $F(t,s)$, then $W_{f,h}$ is the solution of the equation

$$\begin{cases} (\partial_t^2 - \partial_s^2) W_{f,h}(t,s) = F(t,s) & \text{on } \mathbb{R}^+ \times \mathbb{R}^+, \\ W_{f,h}(0,s) = \partial_t W_{f,h}(0,s) = 0. \end{cases}$$

Hence, $W_{f,h}$ is determined by F, and consequently, it is determined by Λ. □

2.5. **Approximate controllability.** The following result is obtained by transposing unique continuation.

Lemma 3 (Approximate controllability)**.** *Let $0 \le s < T \le 1$, then the set*

$$\mathcal{B}(s,T) := \left\{u^f(T,\cdot) : f \in C_0^\infty((T-s,T))\right\}$$

is a dense subset of $L^2(0,s)$.

Note that due to finite speed of propagation the function $u^f(T,\cdot)$ is supported on $[0,s]$. Hence it is natural to view $\mathcal{B}(s,T)$ as a subspace of $L^2(0,s)$.

Proof. To show density, it is enough to prove that $\mathcal{B}(s,T)^\perp = \{0\}$, where

$$\mathcal{B}(s,T)^\perp := \left\{v \in L^2((0,s)) : (v,u)_{L^2((0,s))} = 0 \text{ for all } u \in \mathcal{B}(s,T)\right\}.$$

Let $h \in \mathcal{B}(s,T)^\perp$. Let ω be the solution of the equation

$$\begin{cases} \left(\partial_t^2 - \partial_x^2 + q(t,x)\right)\omega(t,x) = 0 & \text{on } (0,T)\times(0,1); \\ \omega \,|_{x=0,1} = 0; \\ \omega \,|_{t=T} = 0; \\ \partial_t \omega \,|_{t=T} = h. \end{cases}$$

Let $f \in C_0^\infty((T-s,T))$. An integration by parts, together with the boundary and initial conditions for u^f and ω, give

$$0 = \left(\left(\partial_t^2 - \partial_x^2 + q\right) u^f, \omega\right)_{L^2((0,T)\times(0,1))} - \left(u^f, \left(\partial_t^2 - \partial_x^2 + q\right) \omega\right)_{L^2((0,T)\times(0,1))}$$
$$= -\int_0^1 u^f(T,x)\partial_t\omega(T,x)dx - \int_0^T u^f(t,0)\partial_x\omega(t,0)dt.$$

Since $h \in \mathcal{B}(s,T)^\perp$, the first term of the right-hand side is zero, so that

$$0 = \int_0^T f(t)\partial_x\omega(t,0)dt,$$

which is true for arbitrary $f \in C_0^\infty((T-s,T))$. Hence, $\partial_x\omega(t,0) = 0$ on $(T-s,T)$.

Let $\tilde{\omega}$ be the odd extension of ω to $(T,2T)\times(0,1)$, more precisely,

$$\tilde{\omega}(t,x) := \begin{cases} \omega(t,x) & \text{if } t \in t \leq T \\ -\omega(2T-t,x) & \text{otherwise.} \end{cases}$$

Then $\tilde{\omega}$ satisfies

$$\begin{cases} \left(\partial_t^2 - \partial_x^2 + q(t,x)\right)\tilde{\omega}(t,x) = 0 & \text{on } (0,2T)\times(0,1); \\ \tilde{\omega}\mid_{x=0} = 0 & \text{on } (0,2T); \\ \partial_x\tilde{\omega}\mid_{x=0} = 0 & \text{on } (T-s,T+s). \end{cases}$$

By unique continuation Theorem 4, we obtain that $\tilde{\omega} = 0$ on

$$\{(t,x) \in (T-s,T+s)\times(0,1) : |x| \leq s - |T-t|\}.$$

In particular, $\partial_t\omega(T,x) = h(x) = 0$ on $(0,s)$, so that $\mathcal{B}(s,T)^\perp = \{0\}$. □

2.6. **Homework: geometric optics.** We will need the following lemma

Lemma 4. *Let $T \geq 1$, then for any $x_0 \in (0,1)$ there is $f \in C_0^\infty(0,T)$ such that $u^f(T,x_0) \neq 0$.*

Geometric optics can be used to prove the below lemma as outlined in this homework. They also give a method alternative to the Boundary Control method to solve inverse problems[6].

The idea is to find first an approximate solution of the form

$$e^{i\sigma\phi(t,x)}(a_0(t,x) + \sigma^{-1}a_1(t,x) + \sigma^{-2}a_2(t,x) + \dots), \quad \sigma >> 1,$$

and then an actual solution $u = e^{i\sigma\phi}(a_0 + \dots) + r_\sigma$ where the remainder r_σ converges to zero as $\sigma \to \infty$. We will begin with the single term approximation $e^{i\sigma\phi}a_0$ and write $a_0 = a$.

[6]see e.g. /https://www.mv.helsinki.fi/home/lsoksane/leipzig.pdf

2.6.1. *Single term ansatz.* The equation

$$(\partial_t^2 - \partial_x^2 + q)u = 0$$

is equivalent with

$$(\partial_t^2 - \partial_x^2 + q)r_\sigma = -(\partial_t^2 - \partial_x^2 + q)(e^{i\sigma\phi}a), \tag{14}$$

and we want to choose ϕ and a so that

$$(\partial_t^2 - \partial_x^2)(e^{i\sigma\phi}a) = e^{i\sigma\phi}(\partial_t^2 - \partial_x^2)a. \tag{“C”}$$

The rationale is that in this case the absolute value of the right-hand side of (14) is independent from σ, and therefore r_σ is at least not blowing up as $\sigma \to \infty$.

It is a simple matter to expand the left-hand side of (“C”) but a useful computational technique is to consider the conjugated wave operator

$$e^{-i\sigma\phi}(\partial_t^2 - \partial_x^2)e^{i\sigma\phi} = e^{-i\sigma\phi}\partial_t^2 e^{i\sigma\phi} + \cdots = e^{-i\sigma\phi}\partial_t e^{i\sigma\phi} e^{-i\sigma\phi}\partial_t e^{i\sigma\phi} + \ldots.$$

Now, as an operator, $e^{-i\sigma\phi}\partial_t(e^{i\sigma\phi}\cdot) = \partial_t \cdot + i\sigma(\partial_t\phi)\cdot$ and

$$(e^{-i\sigma\phi}\partial_t(e^{i\sigma\phi}\cdot))^2 = \partial_t^2 \cdot + 2i\sigma(\partial_t\phi)\partial_t \cdot - \sigma^2|\partial_t\phi|^2 \cdot + i\sigma(\partial_t^2\phi)\cdot.$$

Treating the spacial derivatives in the same way we get

$$\begin{aligned} e^{-i\sigma\phi}(\partial_t^2 - \partial_x^2)\left(e^{i\sigma\phi}\cdot\right) &= (\partial_t^2 - \partial_x^2)\cdot \\ &+ i\sigma\left(2\partial_t\phi\partial_t \cdot - 2\partial_x\phi\partial_x \cdot + (\partial_t^2 - \partial_x^2)\phi\cdot\right) - \sigma^2\left(|\partial_t\phi|^2 - |\partial_x\phi|^2\cdot\right). \end{aligned}$$

Therefore for $a \neq 0$, (“C”) is equivalent with the following two equations

$$|\partial_t\phi|^2 - |\partial_x\phi|^2 = 0, \tag{E}$$

$$2\partial_t\phi\partial_t a - 2\partial_x\phi\partial_x a + (\partial_t^2\phi - \partial_x^2\phi)a = 0. \tag{T}$$

It is natural to normalize ϕ so that (E) becomes $|\partial_t\phi|^2 = |\partial_x\phi|^2 = 1$. There is some freedom when choosing a solution to (E), but for our purposes it suffices to use the linear solution $\phi(t, x) = t - x$.

The transport equation (T) simplifies now to

$$\partial_t a + \partial_x a = 0.$$

The solutions to this are of the form $a(t, x) = \chi(t - x)$.

2.6.2. *Two-term ansatz.* Let us consider the two term approximation,

$$e^{i\sigma\phi}A, \quad A = a_0 + \sigma^{-1}a_1,$$

and choose ϕ and $a_0 = a$ as above. As we are using a more complicated amplitude, we can ask for more than (“C”), namely

$$(\partial_t^2 - \partial_x^2 + q)(e^{i\sigma\phi}A) = \mathcal{O}(\sigma^{-1}), \quad \sigma >> 1.$$

We use the conjugation formula (E), to obtain

$$e^{-i\sigma\phi}(\partial_t^2 - \partial_x^2 + q)\left(e^{i\sigma\phi}A\right) = (\partial_t^2 - \partial_x^2 + q)A + 2i(\partial_t + \partial_x)a_1.$$

This is of order σ^{-1} whenever a_1 solves the transport equations

$$\partial_t a_1 + \partial_{x^1} a_1 - \frac{i}{2}(\partial_t^2 - \partial_x^2 + q)a_0 = 0, \quad j = 1, 2,$$

or after the change of variables,

$$s = \frac{t + x^1}{2}, \quad r = \frac{t - x^1}{2},$$

equivalently $\partial_s a_1 = \frac{i}{2}(\partial_t^2 - \partial_x^2 + q)a_0$. Therefore we may choose

$$a_1(s, r, x') = \frac{i}{2}\int_{-r}^{s}(\partial_t^2 - \partial_x^2 + q)a_0(s', r, x')ds'.$$

Note that $t = 0$ is equivalent with $s = -r$. The choice of the lower limit $-r$ in the integration implies that $a_1 = 0$, when $t = 0$.

2.6.3. *Solving for the remainder.* When $\chi \in C_0^\infty(\mathbb{R})$ and $\eta \in C_0^\infty(\mathbb{R}^{n-1})$, the restrictions of all the amplitudes a_j, $j = 0, 1, 2$, are compactly supported in $[0, T] \times \mathbb{R}^n$. We recall that the wave equation

$$\begin{aligned} &\Box u + qu = F, \quad \text{in } (0, T) \times \mathbb{R}^n, \\ &u|_{t=0} = \partial_t u|_{t=0} = 0. \end{aligned}$$

has a unique solution u satisfying

$$\|u\|_{C(0,T;H^1(\mathbb{R}^n))} + \|u\|_{C^1(0,T;L^2(\mathbb{R}^n))} \le C\|F\|_{L^2((0,T)\times\mathbb{R}^n)},$$

see e.g. [1, Theorem 7.6]. We solve

$$\begin{aligned} &\Box r_\sigma + qr_\sigma = -(\Box + q)(e^{i\sigma\phi}A), \quad \text{in } (0, T) \times \mathbb{R}^n, \\ &r_\sigma|_{t=0} = \partial_t r_\sigma|_{t=0} = 0. \end{aligned}$$

As the right-hand side is pointwise of order σ^{-1} and compactly supported, we see that $r_\sigma|_{t=T} = \mathcal{O}(\sigma^{-1})$ in $H^1(\mathbb{R}^n)$.

Homework 14. *Prove Lemma 4. Hint: Use the Sobolev embedding $H^1(\mathbb{R}) \subset C(\mathbb{R})$, see [4]. The idea is that r_σ is small pointwise and thus a_0 dominates pointwise along the ray $t = x$ assuming that χ in its definition satisfies $\chi(0) \neq 0$.*

Remark 1. *In the arguments above, we consider two terms approximation, however, we also could derive a longer expansion. This is useful for obtaining better regularity for the solution, which again is important if we consider the higher dimensional case, where the corresponding Sobolev embedding theorem becomes $H^k(\mathbb{R}^n) \subset C(\mathbb{R}^n)$ for $k > n/2$.*

2.7. **Solution to the inverse problem.** Consider two potentials q_1 and q_2. Let us write u_1^f, u_2^f for the solutions of (1) with q replaced by q_1 and q_2, respectively. The corresponding Dirichlet-to-Neumann operators, see (2), are denoted by Λ_1 and Λ_2. We will show that $\Lambda_1 = \Lambda_2$ implies $q_1 = q_2$.

Lemma 5. *Assume that $\Lambda_1 = \Lambda_2$. Let $0 < x < T$ and let*

$$f_j \in C_0^\infty(T-x,T), \quad j = 1,2,\dots,$$

be a sequence such that

$$u_1^{f_j}(T,\cdot) \to 1_{(0,x)}(\cdot) u_1^f(T,\cdot) \qquad \text{in } L^2(0,1). \tag{15}$$

Then

$$u_2^{f_j}(T,\cdot) \to 1_{(0,x)}(\cdot) u_2^f(T,\cdot) \qquad \text{in } L^2(0,1).$$

Proof. Let $\tilde f \in C_0^\infty(T-x,T)$ and let $k = 1,2$. We compute

$$\begin{aligned}\left\|u_k^{\tilde f}(T,\cdot) - u_k^f(T,\cdot)\right\|_{L^2(0,1)}^2 &= \left\|u_k^{\tilde f}(T,\cdot) - 1_{(0,x)}(\cdot)u_k^f(T,\cdot)\right\|_{L^2(0,1)}^2 \\ &+ \left\|\left(1_{(0,x)}(\cdot) - 1\right) u_k^f(T,\cdot)\right\|_{L^2(0,1)}^2 \\ &+ 2\left(u_k^{\tilde f}(T,\cdot) - 1_{(0,x)}(\cdot)u_k^f(T,\cdot), \left(1_{(0,x)}(\cdot) - 1\right) u_k^f(T,\cdot)\right)_{L^2(0,1)}.\end{aligned} \tag{16}$$

Due to finite speed of propagation, we know that $u_k^{\tilde f}(T,\cdot)$ is supported in $(0,x)$. Therefore, the functions

$$u_k^{\tilde f}(T,\cdot) - 1_{(0,x)}(\cdot)u_k^f(T,\cdot) \qquad \left(1_{(0,x)}(\cdot) - 1\right) u_k^f(T,\cdot)$$

have disjoint supports, so that (16) becomes

$$\begin{aligned}\left\|u_k^{\tilde f}(T,\cdot) - u_k^f(T,\cdot)\right\|_{L^2(0,1)}^2 &= \left\|u_k^{\tilde f}(T,\cdot) - 1_{(0,x)}(\cdot)u_k^f(T,\cdot)\right\|_{L^2(0,1)}^2 \\ &+ \left\|\left(1_{(0,x)}(\cdot) - 1\right) u_k^f(T,\cdot)\right\|_{L^2(0,1)}^2.\end{aligned} \tag{17}$$

In particular,

$$\inf_{\tilde f \in C_0^\infty(T-x,T)} \left\|u_k^{\tilde f}(T,\cdot) - u_k^f(T,\cdot)\right\|_{L^2(0,1)}^2 = \left\|\left(1_{(0,x)}(\cdot) - 1\right) u_k^f(T,\cdot)\right\|_{L^2(0,1)}^2. \tag{18}$$

Let $f_j \in C_0^\infty(T-x,T)$ satisfy (15). Then

$$\lim_{j\to\infty} \left\|u_1^{f_j}(T,\cdot) - u_1^f(T,\cdot)\right\|_{L^2(0,1)}^2 = \inf_{\tilde f \in C_0^\infty((T-x,T))} \left\|u_1^{\tilde f}(T,\cdot) - u_1^f(T,\cdot)\right\|_{L^2(0,1)}^2.$$

Due to Lemma 2,

$$\lim_{j\to\infty} \left\| u_2^{f_j}(T,\cdot) - u_2^f(T,\cdot) \right\|^2_{L^2(0,1)} = \inf_{\tilde f \in C_0^\infty((T-x,T))} \left\| u_2^{\tilde f}(T,\cdot) - u_2^f(T,\cdot) \right\|^2_{L^2(0,1)}.$$

Using (17) and (18) we see that

$$\begin{aligned} &\left\| u_2^{f_j}(T,\cdot) - 1_{(0,x)}(\cdot) u_2^f(T,\cdot) \right\|^2_{L^2(0,1)} \\ &\quad = \left\| u_2^{f_j}(T,\cdot) - u_2^f(T,\cdot) \right\|^2_{L^2(0,1)} - \left\| \left(1_{(0,x)}(\cdot) - 1\right) u_2^f(T,\cdot) \right\|^2_{L^2(0,1)} \to 0. \end{aligned}$$

□

Corollary 1. *Assume that* $\Lambda_1 = \Lambda_2$. *Let* $f,\ g \in C_0^\infty(\mathbb{R}^+)$, *let* $T, s \in \mathbb{R}^+$ *and let* $x \in (0,T)$. *Then*

(19)
$$\left(1_{(0,x)}(\cdot) u_1^f(T,\cdot), u_1^h(s,\cdot)\right)_{L^2(0,1)} = \left(1_{(0,x)}(\cdot) u_2^f(T,\cdot), u_2^h(s,\cdot)\right)_{L^2(0,1)}.$$

Proof. By Lemma 3, there is a sequence $\{f_j\} \subset C_0^\infty(T-x,T)$ such that

$$u_k^{f_j}(T,\cdot) \to 1_{(0,x)}(\cdot) u_k^f(T,\cdot)$$

in L^2 for $k=1$. By Lemma 5 this holds also for $k=2$. In view of Lemma 2, we have

$$\begin{aligned} &\left(1_{(0,x)}(\cdot) u_1^f(T,\cdot), u_1^h(s,\cdot)\right)_{L^2(0,1)} = \lim_{j\to\infty} \left(u_1^{f_j}(T,\cdot), u_1^h(s,\cdot)\right)_{L^2(0,1)} \\ &= \lim_{j\to\infty} \left(u_2^{f_j}(T,\cdot), u_2^h(s,\cdot)\right)_{L^2(0,1)} = \left(1_{(0,x)}(\cdot) u_2^f(T,\cdot), u_2^h(s,\cdot)\right)_{L^2(0,1)}. \end{aligned}$$

□

Now we are ready to prove the main result of this section

Theorem 6. *If* $\Lambda_1 = \Lambda_2$, *then* $q_1 = q_2$.

Proof. Assume that $\Lambda_1 = \Lambda_2$, then (19) holds. We consider $T > 1$ and let $x \in (0,1)$. Let us differentiate the left-hand side of (19):

$$\begin{aligned} &\partial_x \left(1_{(0,x)}(\cdot) u_1^f(T,\cdot), u_1^h(s,\cdot)\right)_{L^2((0,1))} \\ &\qquad = \partial_x \int_0^x u_1^f(T,y) u_1^h(s,y)dy = u_1^f(T,x) u_1^h(s,x). \end{aligned}$$

Therefore, by (19), we obtain

(20)
$$u_1^f(T,x) u_1^h(s,x) = u_2^f(T,x) u_2^h(s,x).$$

Due to Lemma 4, for each $x \in (0,1)$ there is $f \in C_0^\infty(\mathbb{R}^+)$ such that $u_1^f(T,x) \neq 0$. Choosing such f we may define

$$w(x) = \frac{u_2^f(T,x)}{u_1^f(T,x)}.$$

We emphasize that the choice of f depends on x, and it may appear that w could be non-smooth. However, it is smooth. Indeed,

$$u_1^h(s,x) = w(x) u_2^h(s,x), \tag{21}$$

and for each $x_0 \in (0,1)$ there is $h \in C_0^\infty(\mathbb{R}^+)$ such that $u_2^h(T,x_0) \neq 0$. Thus, for x near x_0,

$$w(x) = \frac{u_1^h(T,x)}{u_2^h(T,x)}.$$

As the right-hand side is smooth for x near x_0, and as $x_0 \in (0,1)$ is arbitrary, we see that w is smooth.

Let us now take $f = h$ and $s = T$ in (20) and use (21),

$$(u_2^h(T,x))^2 = (u_1^h(T,x))^2 = w^2(x)(u_2^h(T,x))^2.$$

Choosing again $x \in (0,1)$ and $h \in C_0^\infty(\mathbb{R}^+)$ such that $u_2^h(T,x) \neq 0$, we see that $w^2(x) = 1$. The smoothness of w implies that it is a constant function taking the value 1 or -1. To summarize

$$u_1^h(s,x) = \pm u_2^h(s,x),$$

for all $s > 0$, $x \in (0,1)$ and $h \in C_0^\infty(\mathbb{R}^+)$.

There holds

$$0 = (\partial_t^2 - \partial_x^2 + q_1) u_1^h = \pm(\partial_t^2 - \partial_x^2 + q_1) u_2^h = \pm(q_1 - q_2) u_2^h.$$

Choosing such $h \in C_0^\infty(\mathbb{R}^+)$ that $u_2^h(T,x) \neq 0$, we get $q_1(x) = q_2(x)$. $\square$

3. A geometric inverse problem

The main advantage of the Boundary Control method is that it works in general geometric settings. Let (M,g) be a Riemannian manifold with boundary and $q \in C^\infty(M)$. Consider the wave equation

$$\begin{cases} \partial_t^2 u - \Delta_g u + qu = 0 \quad \text{on } \mathbb{R}^+ \times M, \\ u\,|_{x \in \partial M} = f, \\ u\,|_{t=0} = \partial_t u\,|_{t=0} = 0. \end{cases} \tag{22}$$

Here Δ_g is the Laplacian. In local coordinates, it is given as follows

$$\Delta_g \cdot = \frac{1}{\sqrt{\det(g)}} \sum_{i,j=1}^{n} \partial_i \left(\sqrt{\det(g)} g^{i,j} \partial_j \cdot \right).$$

We define the Dirichlet-to-Neumann map

$$\Lambda : C_0^\infty(\mathbb{R}^+ \times \partial M) \to C^\infty(\mathbb{R}^+ \times \partial M)$$

as follows

$$\Lambda f = \partial_\nu u^f \mid_{\mathbb{R}^+ \times \partial M},$$

where u^f is the solution in of (22). We will consider the inverse problem: Determine the potential q given (M, g) and Λ.

It turns out that the speed of propagation is given in terms of the natural distance function d_g on (M, g). That is, it takes the time $d_g(x, y)$ for a wave to propagate from a point $x \in M$ to a point $y \in M$. We recall the definition of the distance on Riemannian manifold.

If (M, g) is is a connected Riemannian manifold and p, $q \in M$, the (Riemannian) distance between p and q, denoted by $d_g(p, q)$, is defined as follows

$$d_g(p, q) := \inf\{l(\gamma) : \gamma \in C_{p,q}\},$$

where

$$C_{p,q} := \{\gamma : [0, b] \to M : \gamma \text{ is a piecewise smooth, continuous curve and } \gamma(0) = p,\ \gamma(b) = q\}$$

and

$$l(\gamma) := \int_0^b \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))} dt.$$

Before entering into the multidimensional geometric case, let us see how non-constant speed of sound gives the speed of propagation in $1 + 1$d.

3.1. **Finite speed of propagation with nonconstant speed of sound.** Let $c \in C^\infty([0, 1])$ and suppose that $c(x) > 0$ for all $x \in [0, 1]$. Consider a solution u to

$$(\partial_t^2 - c^2 \partial_x^2) u = 0 \quad \text{on } \mathbb{R}^+ \times (0, 1), \tag{23}$$

which satisfies

$$u \mid_{x=1} = 0.$$

We set

$$\mathcal{E}(t, x) := c^{-2}(x) |\partial_t u(t, x)|^2 + |\partial_x u(t, x)|^2.$$

We want to find an increasing function $r \in C^\infty(\mathbb{R})$ with $r(0) = 0$ such that the energy

$$E(t) = \frac{1}{2} \int_{r(t)}^1 \mathcal{E}(t, x) dx$$

satisfies

$$\partial_t E(t) \leq 0. \tag{24}$$

The slower r increases, the better finite speed of propagation result we will get.

We write, by using Leibniz integral rule,

$$\begin{aligned}\partial_t E(t) &= -\frac{1}{2} r'(t) \mathcal{E}(t, r(t)) + \frac{1}{2} \int_{r(t)}^1 \partial_t \mathcal{E}(t, x) dx \\ &= -\frac{1}{2} r'(t) \mathcal{E}(t, r(t)) + \int_{r(t)}^1 \left(c^{-2}(x) \partial_t u(t, x) \partial_t^2 u(t, x) + \partial_x u(t, x) \partial_{tx} u(t, x) \right) dx.\end{aligned}$$

Integration by parts gives

$$\begin{aligned}\partial_t E(t) = -\frac{1}{2} r'(t) \mathcal{E}(t, r(t)) &+ [\partial_x u(t, x) \partial_t u(t, x)]_{r(t)}^1 \\ &+ \int_{r(t)}^1 \left(c^{-2} \partial_t^2 u(t, x) + \partial_x^2 u(t, x) \right) \partial_t u(t, x) dx.\end{aligned}$$

Since u is the solution of the wave equation the last integral is 0. Moreover, due to the boundary condition, the second term is 0 at $x = 1$, so that

$$\partial_t E(t) = -\frac{1}{2} r'(t) \mathcal{E}(t, r(t)) - \left. (\partial_t u(t, x) \partial_x u(t, x)) \right|_{x=r(t)}.$$

Note that $r' > 0$ since r is increasing. Therefore, using a simple inequality

$$2\sqrt{xy} \leq \alpha x^2 + \frac{1}{\alpha} y^2, \qquad \text{for } \alpha, x, y > 0,$$

we know that (24) holds if

$$\frac{1}{r'(t)} = r'(t) c^{-2}(r(t)),$$

or equivalently,

$$r'(t) = c(r(t)). \tag{25}$$

This equation is solvable. Indeed, consider the following function

$$\rho(x) = \int_0^x \frac{1}{c(y)} dy.$$

Since ρ is a strictly increasing function, its inverse function exists, so we can set

$$r(t) = \rho^{-1}(t)$$

It is easy to check that this function indeed satisfies (25). Observe that if $c = 1$ identically, we obtain $r(t) = t$.

The function ρ is the travel time between 0 and x: the time necessary for perturbation at 0 to reach x. It also can be interpreted as the distance from 0 to x. Indeed, let us consider the metric

$$g = c^{-2}dx,$$

on $[0, 1]$. Then, for $x \in [0, 1]$, we get

$$d_g(0, x) = \inf_{\gamma \in C_{0,x}} \int_0^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}dt.$$

Let γ be a minimizer curve. Then $\gamma : [0, b] \to [0, x]$ needs to be a bijection. By changing the coordinates $\tau = \gamma(t)$, we obtain

$$d_g(0, x) = \int_0^b \dot{\gamma}(t)\frac{1}{|c(\gamma(t))|}dt = \int_0^x \frac{1}{|c(\tau)|}d\tau = \rho(x).$$

3.2. **Main tools.** As in the one dimensional case, the main ingredients of solving the inverse problem we are considering here are the finite speed of propagation and unique continuation.

Theorem 7 (Finite speed of propagation, nD)**.** *Let $\Omega \subset M$ be an open set, and define*

$$\mathcal{C} := \{(t, x) \in \mathbb{R} \times M : d_g(x, M \setminus \Omega) > |t|\}.$$

Assume that u is a solution of the equation

$$\begin{cases} (\partial_t^2 - \Delta_g + q(x))\, u = 0, & \text{on } \mathbb{R} \times M; \\ u\,|_{t=0} = \partial_t u\,|_{t=0} = 0, & \text{on } \Omega. \end{cases}$$

Assume, furthermore, that $u|_{\mathcal{C}\cap(\mathbb{R}\times\partial M)} = 0$. Then $u|_{\mathcal{C}} = 0$.

We omit the proof and refer to [3, Theorem 2.47].

Theorem 8 (Unique continuation, nD)**.** *Let $u \in H^1([-T, T] \times M)$ satisfy the equation*

$$\partial_t^2 u - \Delta_g u + qu = 0 \quad \text{on } \mathbb{R}^+ \times M.$$

Assume that

$$u\,|_{[-T,T]\times\Gamma} = \partial_\nu u\,|_{[-T,T]\times\Gamma} = 0,$$

where $\Gamma \subset \partial M$ is open. Then

$$u\,|_K = 0,$$

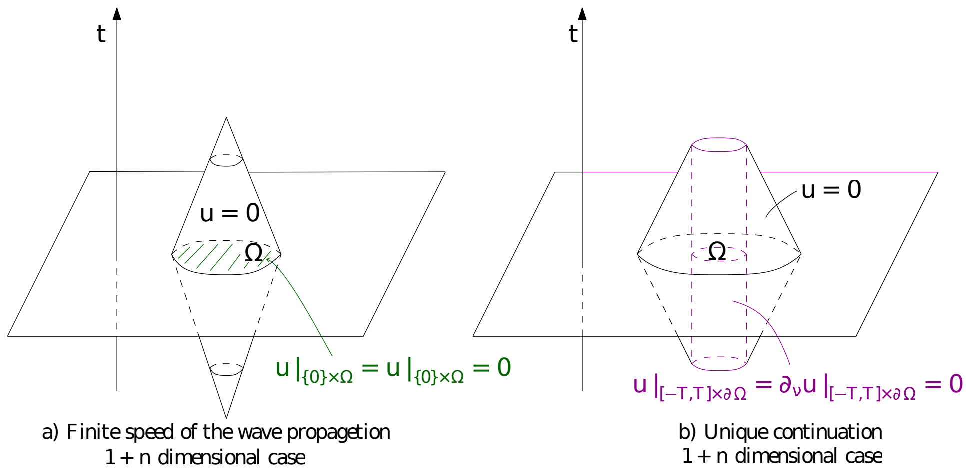

FIGURE 2.

where

$$K := \{(t,x) \in [-T,T] \times M : d_g(x,\Gamma) \leq T - |t|\}.$$

We omit the proof and refer to [3, Theorem 3.16].

Applying Theorem 8 on $M \setminus \Omega$, where $\Omega \subset M$ is a small open set with smooth boundary satisfying $\partial\Omega \cap \partial M = \emptyset$, allows us to compare unique continuation to finite speed of propagation, see Figure 2. Clearly these two results are genuinely different in the higher dimensional case, in contrast to the one dimensional case. (Recall that in the dimension one, finite speed of propagation and unique continuation differ only by interchanging space and time.)

Let $\Gamma \subset \partial M$ be an open set. We define the domain of influence

$$M(\Gamma, T) := \{x \in M : d_g(x,\Gamma) \leq T\}$$

and the set

$$L^2(M(\Gamma,T)) := \{\phi \in L^2(M) : \text{supp}(\phi) \subset M(\Gamma,T)\}$$

equipped with the L^2 norm.

Similarly, as we derived Lemma 3 from Theorem 4, Theorem 8 gives us the following approximate controlability:

Lemma 6. *Let $T > 0$, then the set*

$$\mathcal{B}(T,\Gamma) := \{u^f(T,\cdot) : f \in C_0^\infty((0,T) \times \Gamma)\}$$

is a dense subset of $L^2(M(\Gamma,T))$.

Homework 15. *Prove the theorem above.*

Lemma 4 can be generalized for the multidimensional case:

Lemma 7. *Let $T > 0$, then for any point x_0 in the interior of $M(\Gamma, T)$ there is $f \in C_0^\infty((0,T) \times \Gamma)$ such that $u^f(T, x_0) \neq 0$.*

Homework 16. *Prove the lemma above.*

Remark 2. *To obtain the lemma above, higher regularity estimates and longer ansatz are needed to get hold of point values, since more regularity is needed in higher dimensional Sobolev embedding.*

3.3. **Solution to the inverse problem.** Let $T > 0$. We define for functions f, $h \in C_0^\infty(\mathbb{R}^+ \times \partial M)$

$$W_{f,h}(t,s) = (u^f(t,\cdot), u^h(s,\cdot))_{L^2(M)}.$$

The following lemma is the higher dimensional analogue of Lemma 2.

Lemma 8. *Let $T > 0$ and f, $h \in C_0^\infty(\mathbb{R}^+ \times \partial M)$. Then the Dirichlet-to-Neumann map Λ determines $W_{f,h}$.*

Proof. We write

$$\begin{aligned}(\partial_t^2 - \partial_s^2) W_{f,h}(t,s) &= \left(\Delta_g u^f(t,\cdot) - q u^f(t,\cdot), u^h(s,\cdot)\right)_{L^2(M)} \\ &\quad - \left(u^f(t,\cdot), \Delta_g u^h(s,\cdot) - q u^h(s,\cdot)\right)_{L^2(M)}.\end{aligned}$$

Further, by Green's identity, we obtain

$$\begin{aligned}(\partial_t^2 - \partial_s^2) W_{f,h}(t,s) &= \int_{\partial M} \partial_\nu u^f(t,x) u^h(s,x) dS(x) - \int_{\partial M} u^f(t,x) \partial_\nu u^h(s,x) dS(x) \\ &= (\Lambda f(t,\cdot), h(s,\cdot))_{L^2(\partial M)} - (f(t,\cdot), \Lambda h(s,\cdot))_{L^2(\partial M)}.\end{aligned}$$

Let us denote the right-hand side by $F(t,s)$, then $W_{f,h}$ is the solution of the equation

$$\begin{cases}(\partial_t^2 - \partial_s^2) W_{f,h}(t,s) = F(t,s) & \text{on } (0,T) \times (0,T), \\ W_{f,h}(0,s) = \partial_t W_{f,h}(0,s) = 0. & \end{cases}$$

Hence, $W_{f,h}$ is determined by F, and consequently, it is determined by Λ. □

We have the analogue of Lemma 5.

Lemma 9. *Assume that $\Lambda_1 = \Lambda_2$. Let $0 < s < T$, let $\Gamma \subset \partial M$ be open and let $\{f_j\} \subset C_0^\infty((T-s,T) \times \Gamma)$ be a sequence such that*

$$u_1^{f_j}(T,\cdot) \to 1_{M(\Gamma,s)}(\cdot) u_1^f(T,\cdot) \qquad \text{in } L^2(M).$$

Then

$$u_2^{f_j}(T,\cdot) \to 1_{M(\Gamma,s)}(\cdot) u_2^f(T,\cdot) \qquad \text{in } L^2(M).$$

We omit the proof as it coincides with that in the one dimensional case. As before, we have also the following corollary

Corollary 2. *Assume that* $\Lambda_1 = \Lambda_2$. *Let* $0 < s < T$, $t > 0$ *and let* $\Gamma \subset \partial M$ *be open, then for any* f, $h \in C_0^\infty(\mathbb{R}^+ \times \partial M)$, *it follows*

$$\left(1_{M(\Gamma,s)}(\cdot) u_1^f(T,\cdot), u_1^h(t,\cdot)\right)_{L^2(M)} = \left(1_{M(\Gamma,s)}(\cdot) u_1^f(T,\cdot), u_1^h(t,\cdot)\right)_{L^2(M)}.$$

Now we deviate from the one dimensional proof.

Corollary 3. *Assume that* $\Lambda_1 = \Lambda_2$. *Let* $s, \tilde{s} \in (0,T)$, *let* $t > 0$ *and consider two open* Γ, $\tilde{\Gamma} \subset \partial M$. *Then we have for any* f, $h \in C_0^\infty(\mathbb{R}^+ \times \partial M)$

$$\left(1_{M(\Gamma,s)} 1_{M(\tilde{\Gamma},\tilde{s})} u_1^f(T,\cdot), u_1^h(t,\cdot)\right)_{L^2(M)} \\ = \left(1_{M(\Gamma,s)} 1_{M(\tilde{\Gamma},\tilde{s})} u_1^f(T,\cdot), u_1^h(t,\cdot)\right)_{L^2(M)}.$$

Proof. By Lemmas 6 and 9, there is a sequence $\{f_j\} \subset C_0^\infty((T-s,T) \times \Gamma)$ such that

$$u_1^{f_j}(T,\cdot) \to 1_{M(\Gamma,s)} u_1^f(T,\cdot) \quad \text{and} \quad u_2^{f_j}(T,\cdot) \to 1_{M(\Gamma,s)} u_2^f(T,\cdot).$$

Therefore, using Corollary 2, we obtain

$$\begin{aligned} &\left(1_{M(\Gamma,s)} 1_{M(\tilde{\Gamma},\tilde{s})} u_1^f(T,\cdot), u_1^h(t,\cdot)\right)_{L^2(M)} \\ &= \lim_{j\to\infty} \left(1_{M(\tilde{\Gamma},\tilde{s})} u_1^{f_j}(T,\cdot), u_1^h(t,\cdot)\right)_{L^2(M)} \\ &= \lim_{j\to\infty} \left(1_{M(\tilde{\Gamma},\tilde{s})} u_2^{f_j}(T,\cdot), u_2^h(t,\cdot)\right)_{L^2(M)} \\ &= \left(1_{M(\Gamma,s)} 1_{M(\tilde{\Gamma},\tilde{s})} u_1^f(T,\cdot), u_1^h(t,\cdot)\right)_{L^2(M)}. \end{aligned}$$

□

Theorem 9. *If* $\Lambda_1 = \Lambda_2$, *then* $q_1 = q_2$.

Proof. Let x_0 be an interior point of M and $s := d_g(x_0, \partial M)$. We choose $T > s$. Let $y \in \partial M$ such that

$$d_g(x,y) = s.$$

Let Γ, $\tilde{\Gamma} \subset \partial M$ be open subsets and $y \in \tilde{\Gamma} \subset \Gamma$. In fact, we can take $\Gamma = \partial M$. Let also $\tilde{s} > s$ and set

$$Z := M(\tilde{\Gamma}, \tilde{s}) \setminus M(\Gamma, s).$$

Then,

$$\frac{1}{|Z|}\left(\left(1_{M(\tilde{\Gamma},\tilde{s})}u_1^f(T,\cdot),u_1^h(t,\cdot)\right)_{L^2(M)}-\left(1_{M(\tilde{\Gamma},\tilde{s})}1_{M(\Gamma,s)}u_1^f(T,\cdot),u_1^h(t,\cdot)\right)_{L^2(M)}\right) \to u_1^f(T,x_0)u_1^h(t,x_0)$$

as $\tilde{s} \to s$ and $\tilde{\Gamma} \to \{y\}$. The same holds for u_2^f and u_2^h, so that by Corollaries 2 and 3, we know that

$$u_1^f(T,x_0)u_1^h(t,x_0) = u_2^f(T,x_0)u_2^h(t,x_0).$$

This is the analogue of (20), and we conclude as in the proof of Theorem 6. □

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