JSS 31 - MA2 - Summer '22
Introduction to the mathematics of X-ray imaging: X-ray transforms

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August 14, 2022

Abstract

These lecture notes are associated with the 5-day mini-course given by the author at the 31st summer school at the University of Jyväskylä, Aug. 8-12, 2022. After dicsussing generalities associated with inverse problems such as injectivity, stability and mapping properties, we discuss three prototypes of integral-geometric problems in special, symmetric cases (the Funk transform on the 2-sphere; the Radon transform on the plane; the X-ray transform on the unit disk) and attempt a thorough analysis of these problems via Fourier methods. These three cases differ by their compactness or non-compactness, and by the presence or absence of a (convex) boundary. Each of these peculiarities is reflected in the analysis, and it is the author's hope that the ideas to approach them may shed some light on how to approach more general situations without symmetries.

Please email any comments/typos/suggestions.

Contents

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1 Lecture 1 - Introduction

Though we will later focus on inverse problems in integral geometry, these problems sit inside the larger field of inverse problems, which prescribes an agenda of questions that one may address. Some examples may then not be directly related to integral geometry, though they are part of the inverse problems folklore (some of them still open to this day).

Other useful references: [Bal12, Chapter 1], [Ilm17, §1]

1.1 What is an inverse problem ?

The main question one is interested to address is:

Let $\mathcal{M}\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map between two topological spaces $\mathcal{X}$, $\mathcal{Y}$.
Given $y \in \mathcal{Y}$, find $x \in \mathcal{X}$ such that $\mathcal{M}(x) = y$.

As general comments:

- $\mathcal{M}$ can be linear or non-linear.
- $\mathcal{X}$ and $\mathcal{Y}$ can be finite-dimensional normed spaces or, most often, infinite-dimensional topological vector spaces (Banach, Hilbert, Fréchet spaces of functions and distributions) or topological manifolds.
- The spaces above often reflect a notion of **smoothness**, typically embodied by a scale of Hilbert or Banach spaces (e.g. Sobolev spaces), whose intersection gives a Fréchet space (e.g. smooth functions, rapidly-decaying functions or both), or LF spaces (e.g. smooth functions with compact support). A **scale** of spaces is a family of Banach or Hilbert spaces $\{(H^s, \|\cdot\|_s\}_{s\in\mathbb{N}_0}$ with continuous injections $H^s \hookrightarrow H^t$, $s \geq t$, and the intersection may be given a Fréchet topology defined by the countable family of seminorms $\|\cdot\|_s$, $s \in \mathbb{N}_0$. Important examples include Sobolev spaces ($f \in H^k(\mathbb{R})$ iff $\partial^\alpha f \in L^2$ for all $0 \leq \alpha \leq k$, with norm $\|f\|^2_{H^k} = \sum_{\alpha=0}^k \|\partial^\alpha f\|^2_{L^2}$) and the classical $C^k(\mathbb{R})$ Banach spaces (equipped with the sup norm $\|f\|_{C^k} := \max_{0\leq\alpha\leq k} \sup_{x\in\mathbb{R}} |\partial^\alpha f(x)|$), with intersection space $C^\infty(\mathbb{R})$. Another way to quantify smoothness is by means of decay in the Fourier domain, which may motivate other scales of spaces of the form $B^k = \{f \in L^2(\mathbb{R}); (1 + x^2)^{k/2} f \in L^2(\mathbb{R})\}$, $k \geq 0$.
- $\mathcal{M}$ may be not injective and not surjective. What, then, does it mean, to 'invert' $\mathcal{M}$?
- It could be that the problem may be formulated as "recover x from $\mathcal{M}(x)$", and part of the job is to find a 'good' pair of spaces $\mathcal{X}$, $\mathcal{Y}$ as well.

Example 1.
- *Finite-dimensional, linear inverse problems (given $A \in \mathbb{C}^{p\times q}$ and $y \in \mathbb{C}^p$, solve $Ax = y$ for $x \in \mathbb{C}^q$).*

- *Inverse diffusion on a ring: to recover the initial temperature distribution $u_0(\theta)$ from the observation of the temperature distribution at a later time $u_T(\theta) = u(\theta, T)$, where $u(\theta, t)$ solves the heat equation*

$$\partial_t u = \partial_\theta^2 u, \qquad \theta \in \mathbb{S}^1, \quad t > 0 \qquad (\textit{heat equation})$$
$$u(\theta, 0) = u_0(\theta). \qquad (\textit{initial condition})$$

  This gives rise to a bounded operator

$$\mathcal{M}\colon L^2(\mathbb{S}^1) \ni f \mapsto u \in L^2(\mathbb{S}^1). \tag{1}$$

- *Bessel's equation on a ring: to recover $f \in L^2(\mathbb{S}^1)$ from the unique solution u of*

$$-\partial_\theta^2 u + u = f \quad (\textit{on } \mathbb{S}^1).$$

  This gives rise to a bounded operator

$$\mathcal{M}\colon L^2(\mathbb{S}^1) \ni f \mapsto u \in L^2(\mathbb{S}^1). \tag{2}$$

- *Funk/X-ray/Radon transforms (see next lectures for details).*

- *(geodesic X-ray transform) A generalization of the X-ray transform is to consider a transform that integrates a function along a family of curves. One way to realize this is by considering M a domain in $\mathbb{R}^2$ and equip it with Riemannian metric giving rise to a family of "geodesic curves". Then the geodesic X-ray transform I_0 is the map that sends a function f defined on M to the collection of its integrals along each geodesics. The function $I_0 f$ is then a function on G, the set of all geodesics through M. The latter space may require geometric assumptions in order to even make sense as a manifold, and once this is done, defining appropriate function spaces on M ang G that accurately capture the mapping properties of I_0 is a task that remains under active study.*

- *(Boundary rigidity) The previous transform is a linearized version of a non-linear inverse problem which consists of reconstructing a Riemannian metric on a manifold-with-boundary, from its so-called boundary distance function. This is called the boundary rigidity problem, also a hard problem under active study. For more info on the previous two examples, see [IM19, PSU22].*

1.2 The inverse problems agenda

Two practically motivated philosophical points may help approach what follows:

- The forward operator $\mathcal{M}$ is generally 'smoothing' and therefore, undoing that process will require 'unsmoothing' the data, a process which is in itself ill-conditioned (or unstable) in the sense that $\mathcal{M}^{-1}\colon \mathcal{Y} \to \mathcal{X}$ may not be continuous in general. If $\mathcal{M}$ is thought of as a linear operator between Hilbert spaces, $A\colon X_0 \to Y_0$, this corresponds to saying that A is **compact**.

- The data y may be corrupted by noise, in the sense that we want to find $x \in \mathcal{X}$ measuring $\mathcal{M}(x) + \eta$, where η models a noise realization which we have little control over. Two issues naturally arise out of this: first, the measurement $\mathcal{M}(x) + \eta$ might no longer live in $\mathcal{Y}$ (depending on how smoothing the problem is, $\mathcal{M}(x)$ may be smoother than η, so $\mathcal{M}(x) + \eta$ lives in the space where η lives, not where $\mathcal{M}(x)$ lives); second, $\mathcal{M}(x) + \eta$ might not even be in the range of $\mathcal{M}$!

With that in mind, here are some of the important questions to be addressed:

1.2.1 Injectivity, and modding out "obvious" obstructions to it

Does $\mathcal{M}(x)$ characterize x uniquely (equivalently, does $\mathcal{M}(x_1) = \mathcal{M}(x_2)$ imply $x_1 = x_2$) ? If not, can one describe obvious obstructions to injectivity ?

There may be ways that the problem is "obviously non-injective", and should first rule out the obvious obstructions. A typical example of this is when there is a "gauge" group $\mathcal{G}$ acting on $\mathcal{X}$ such that for all $x \in \mathcal{X}$ and for all $g \in \mathcal{G}$, we have $\mathcal{M}(g \cdot x) = \mathcal{M}(x)$. As there is no hope to recover more than the equivalence class of x, one should study the new operator

$$\widetilde{\mathcal{M}} \colon \mathcal{X}/\mathcal{G} \to \mathcal{Y}.$$

The quest for injectivity is reformulated as a quest for "injectivity-modulo-gauge": do we have $\mathcal{M}(x_1) = \mathcal{M}(x_2)$ if and only if $x_1 = g \cdot x_2$ for some $g \in \mathcal{G}$?

Example 2.
1. *For a linear operator A, $\mathcal{G} = \ker A$ is one such example provided that one can understand that space.*
2. *In the case of the boundary rigidity problem, $\mathcal{G} = \mathit{Diff}_{\partial M}(M)$, the group of self-diffeomorphisms of M fixing the boundary of M.*

1.2.2 Range characterization

The operator $\mathcal{M} : \mathcal{X} \to \mathcal{Y}$ is most likely not surjective, so how does $\mathcal{M}(\mathcal{X})$ sit inside $\mathcal{Y}$? Given $y \in \mathcal{Y}$, can one find "consistency conditions" which imply that y is in the range of $\mathcal{M}$? If $\mathcal{M}$ is linear, then $\mathcal{M}(\mathcal{X})$ is a linear subspace of $\mathcal{Y}$; can one easily describe the supplementary (sometimes, orthocomplement) of $\mathcal{M}(\mathcal{X})$?

Example 3. *Consider the operator $A \colon \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ given by $A\mathbf{u} = \left(\frac{1-(-1)^n}{n+1} u_n\right)_n$, $\mathbf{u} = (u_n)_n$. Equip $\ell^2(\mathbb{N}_0)$ with the orthonormal family $\{\mathbf{e}_n\}_{n\geq 0}$ where for $n \geq 0$, $\mathbf{e}_n = \{\delta_{jn}\}_{j\geq 0}$.*

A is not surjective for two fundamentally different reasons: any basis vector $\mathbf{e}_{2k}$ is not achieved in the range of A, and if we were to restrict A to the span $\langle \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5, \dots \rangle$, where it becomes injective, it is still not surjective because elements in the range have faster decay than ℓ^2. In particular, the sequence $\mathbf{u} = \left\{\frac{1+(-1)^n}{n}\right\}_n \in \ell^2(\mathbb{N}_0)$ seems to have a preimage by A, but that preimage does not belong to $\ell^2(\mathbb{N}_0)$.

1.2.3 Stability

Suppose injectivity has been settled. Stability is a quantification of how 'well-behaved' the inversion process will be.

Characterization via moduli of continuity. One way to think of stability is to ask what is the form of the modulus of continuity of $\mathcal{M}^{-1} : \mathcal{Y} \to \mathcal{X}$, if any ? i.e., for which function $\omega : [0, \infty) \to [0, \infty)$ with $\lim_{x\to 0} \omega(x) = 0$ do we have[1]

$$\|\mathcal{M}^{-1}(y_1) - \mathcal{M}^{-1}(y_2)\|_{\mathcal{X}} \lesssim \omega(\|y_1 - y_2\|_{\mathcal{Y}}).$$

Since $\mathcal{M}^{-1}$ might not even make sense, we prefer writing

$$\|x_1 - x_2\|_{\mathcal{X}} \lesssim \omega(\|\mathcal{M}(x_1) - \mathcal{M}(x_2)\|_{\mathcal{Y}}). \tag{3}$$

Equation (3) is a *stability estimate*, which quantifies how error on data (measured with $\|\cdot\|_{\mathcal{Y}}$) translates into error on the reconstruction (measured with $\|\cdot\|_{\mathcal{X}}$). It helps answer the question: suppose I want a reconstruction error no greater than ε, what precision on my measurements do I need ?

The problem is then said *Lipschitz-stable in* $(\mathcal{X}, \mathcal{Y})$ if one has equality (3) with $\omega(x) = x$, the best-case scenario[2]; *Hölder-stable in* $(\mathcal{X}, \mathcal{Y})$ if $\omega(x) = x^\alpha$ for some $0 < \alpha < 1$; worse moduli of continuity include $\omega(x) = \frac{1}{|\log x|}$ (*log-stable*, or exponentially ill-posed).

A problem may be made Lipschitz-stable for some specific choice of norm, in spite of the fact that it is objectively badly-behaved. But then, this Lipschitz estimate is probably practically unusable because the space $\mathcal{Y}$ is too small for the noise to live in it. Although we mentioned above that there is some leeway in choosing the spaces $\mathcal{X}$ and $\mathcal{Y}$, a first constraint is to use a space $\mathcal{Y}$ where the noise lives.

Characterization via scales of spaces. Another way to define stability is with respect to Hilbert scales $\mathcal{X}_k$ and $\mathcal{Y}_k$ such that $\mathcal{M} : \mathcal{X}_k \to \mathcal{Y}_k$ is continuous for all k. Then the inverse problem is, relative to this choice of scales,

- *well-posed* if $\|x - x'\|_k \lesssim \|\mathcal{M}(x) - \mathcal{M}(x')\|_k$ for all k
- *mildly ill-posed of order* $\alpha > 0$ if there is $\alpha > 0$ such that $\|x - x'\|_k \lesssim \|\mathcal{M}(x) - \mathcal{M}(x')\|_{k+\alpha}$ for all k. One then seeks the smallest α and calls it the *order of ill-posedness of* $\mathcal{M}$ (it depends on the choice of scales)
- *severely ill-posed* otherwise.

When the grading of the Hilbert scales at play describes order of differentiability, the α above quantifies by how many derivatives the operator $\mathcal{M}$ is smoothing (as a result, reconstructing x will

[1] We will write $a \lesssim b$ if there is a constant C such that $a \le Cb$. Unless important, we will not keep track of constants.

[2] Recall from one-variable calculus: a function with superlinear modulus of continuity is constant.

involve differentiating the data α times). A severely ill-posed problem typically corresponds to an operator which is smoothing by an infinite degree.

What is the link between the above two characterizations of stability ? When the second one is well-understood, one can cook up appropriate moduli of continuity for the inverse, provided that one adds a prior smoothness assumption on the unknown x (see Exercise 5).

Relating scales of spaces. Given a compact, injective and self-adjoint operator B on a Hilbert space $(X_0, (\cdot,\cdot)_0)$, there is a canonical way of constructing a Hilbert scale for which B is exactly ill-posed of order 1. The construction is done as follows:

- By the spectral theorem for compact self-adjoint operators ([Fol95, Th. 0.44]), there exists a complete orthonormal set $\{\mathbf{e}_n\}_n$ of X_0 and non-negative numbers $\sigma_0 \ge \sigma_1 \ge \sigma_2 \ge \dots$, each with finite multiplicity and with $\lim_{n\to\infty} \sigma_n = 0$, such that for all $x \in X_0$,

$$B\mathbf{x} = \sum_{n\ge 0} \sigma_n \ (\mathbf{e}_n, \mathbf{x})_0 \ \mathbf{e}_n$$

- Upon defining the space $X := \cap_{k\ge 0} B^k X_0$, on such elements, one may define the operator B^s for any $s \in \mathbb{R}$, by defining

$$B^s\mathbf{x} := \sum_{n\ge 0} \sigma_n^s \ (\mathbf{e}_n, \mathbf{x})_0 \ \mathbf{e}_n.$$

  Then for every $k \in \mathbb{N}_0$, define the space

$$X_k: \text{ the completion of } X \text{ for the inner product } (\mathbf{x},\mathbf{y})_s := (B^{-s}\mathbf{x}, B^{-s}\mathbf{y})_0, \tag{4}$$

  a Hilbert space by construction. X is then typically equipped with the weakest topology making the injections $X \to X_k$ continuous (i.e., $\mathbf{x}_n$ converges to $\mathbf{x}$ in E if and only if $\|\mathbf{x}_n - \mathbf{x}\|_k \to 0$ as $n \to \infty$ for all k).

- Then it's easy to see that $B\colon X_k \to X_{k+1}$ is an isometry for all k, in particular bounded with bounded inverse. As such, relative to the scale $\{X_k\}_k$, the operator B is ill-posed of order 1.

In some sense, this construction, while being perfectly adapted to describing mapping properties of B, does not necessarily tell us anything about the operator B, as it is unrelated to more standard scales which encode, for instance, derivatives. Part of the work in assessing stability properties of the operator B is then to find *how to relate the scale constructed above to more standard ones* (C^k, Sobolev types, or modelled after specific differential operators), if possible at all.

1.2.4 Further questions

While we will probably not address these aspects here, it is worth noting that the inverse problems community is heavily concerned with the following questions.

Reconstruction: What are the ways that we can recover x from $\mathcal{M}(x)$? (explicit reconstruction formulas, Fredholm equations, regularized inversions, Markov Chain Monte Carlo)

Partial Data problems: What if we only have partial knowledge of $\mathcal{M}(x)$ (e.g. discrete samples, or a restriction of the full data) ? How are injectivity, stability and reconstruction impacted ?

Parameter dependence: Similarly to the previous question: how do the answers to injectivity, stability and reconstruction depend on some parameter in the system considered ? Examples:

- Injectivity and stability of the geodesic X-ray transform on a Riemannian surface depend on whether that surface has conjugate points.
- Injectivity of the attenuated X-ray transform over vector fields degenerates as the attenuation vanishes.
- In an inverse wave problem (TAT/PAT), injectivity and stability depend on the observation time.

Practical questions: What is the nature of the noise in the measurement ? (in what space does it live ?) how to use the stability estimate to understand how errors will magnify ? How much do we have to regularize the inversion in order to obtain a meaningful reconstruction ?

1.3 Some prototypes

- Finite-dimensional, linear inverse problems. Singular Value Decomposition. Recall that for a linear operator $A\colon \mathbb{C}^p \to \mathbb{C}^q$, setting $r = \min(p,q)$, there exists orthonormal bases $(u_1, \dots, u_p)$ and $(v_1, \dots, v_q)$ and non-negative numbers $\sigma_1, \dots, \sigma_r$ such that

  $$Au_j = \sigma_j v_j, \quad A^* v_j = \sigma_j u_j, \qquad 1 \le j \le r.$$

  If $p > q$, then we also have $Au_j = 0$ for $q < j \le p$ and if $q > p$, we have $A^* v_j = 0$ for $p \le j < q$. The u_j's are the eigenvectors of $A^*A\colon \mathbb{C}^p \to \mathbb{C}^p$ (a symmetric operator), the v_j's are the eigenvectors of $AA^*\colon \mathbb{C}^q \to \mathbb{C}^q$, and σ_j^2 are the eigenvalues of either operator.

  In the case where $p = q$ and all singular values are non-zero and arranged in decreasing order $\sigma_1 \ge \dots \ge \sigma_p$, then we have

  $$\sigma_p \|x\| \le \|Ax\| \le \sigma_1 \|x\|, \qquad \forall x \in \mathbb{C}^p, \tag{5}$$

  which gives us both continuity and stability constants.

- Infinite-dimensional linear inverse problems involving compact[3] operators. Suppose A is now a linear, bounded operator between two Hilbert spaces $A\colon \mathcal{H}_1 \to \mathcal{H}_2$. Can we use the SVD picture again ? Well, not always. The operator $A^*A\colon \mathcal{H}_1 \to \mathcal{H}_1$ will be bounded, self-adjoint,

[3] A linear operator between Hilbert spaces $A\colon \mathcal{H}_1 \to \mathcal{H}_2$ is *compact* if it maps bounded sequences to sequences with convergent subsequences.

but its spectrum may not be discrete[4]. However, if A^*A is *compact*, then indeed, we can find two Hilbert orthonormal bases u_n, v_n of $\mathcal{H}_1, \mathcal{H}_2$ and a decreasing sequence of non-negative numbers σ_j such that

$$Au_j = \sigma_j v_j, \qquad A^* v_j = \sigma_j u_j, \qquad j \ge 0.$$

This is the spectral theorem for self-adjoint, compact operators[5]. Moreover, since A^*A is compact, the sequence σ_j necessarily decreases to zero, so $\inf_j \sigma_j = 0$ and there will be no room to write a stability estimate as in (5). Having an SVD is however extremely relevant to understand stability, inversion and regularization purposes.

- Consider the measurement operator as follows: Given $f \in L^2([0,\pi])$, Let $\mathcal{M}(f) = u|_{t=T}$, where $u(x,t)$ solves the heat equation

$$\partial_t u = \partial_{xx} u \qquad (0,\pi)\times(0,T), \qquad u|_{t=0} = f,$$

with Neuman boundary conditions $\partial_x u(0,t) = \partial_x u(\pi,t) = 0$.

1. Expand f and u as Fourier cosine series to make appear an explicit form of the measurement operator. In the Fourier cosine series identification $f \leftrightarrow \{a_n\}_{n\ge 0}$ such that $f(x) = \sum_{n=0}^\infty a_n \cos(nx)$, the operator $\mathcal{M}$ is diagonal and its action looks like $a_n \mapsto a_n e^{-n^2 T}$. In particular, the operator $\mathcal{M}$ (or, rather $\mathcal{F}\mathcal{M}\mathcal{F}^{-1}$ with $\mathcal{F}$ the cosine series transform) is bounded and injective from $\ell^2(\mathbb{N}_0)$ into itself.
2. On the other hand, the inverse is not $\ell^2 \to \ell^2$ continuous, nor is it $h^p \to \ell^2$ bounded for any $p \ge 0$ (study the ratio $\frac{\|f\|_{\ell^2}}{\|\mathcal{M}(f)\|_{h^p}}$ with $f(x) = \cos(nx)$). This is an example of an exponentially ill-posed problem.
3. Yet we can still find a space $\mathcal{H}$ where $\mathcal{M} : \ell^2 \to \mathcal{H}$ is an isometry (i.e. with bounded inverse in particular). This space can be easily found to be

$$\mathcal{H} = \left\{ (a_n) \in \ell^2, \qquad \sum_{n\ge 0} |a_n|^2 e^{2n^2 T} < \infty \right\}.$$

   It corresponds to Fourier series which decay at an exponential rate. For such series, the corresponding Fourier cosine series $\sum_{k=0}^\infty a_k \cos(kx)$ is smooth on $[0,\pi]$. Then $\mathcal{M}$ maps rough functions into smooth ones. Undoing that process would then require "differentiating infinitely many times", one of the interpretations of severe ill-posedness.

1.4 Next lectures...

For the 'friendliest' cases of X-ray transforms on surfaces, one usually says that they are 'smoothing of order 1/2' and their associated normal operators are 'smoothing of order 1'. This section aims at making these statements most explicit, following one or more of the following routes:

[4] See the spectral theorem for bounded, self-adjoint operators

[5] [Fol95, Th. 0.44 + Th. 0.38]: If A is a compact self-adjoint operator on a Hilbert space $\mathcal{H}$, then $\mathcal{H}$ has an orthonormal basis consising of eigenvectors for A. In addition, every non-zero eigenvalue has finite multiplicity, and the the only possible limit point is 0.

- by computing an explicit singular value decomposition of the operator. Once this is done, the construction of Hilbert scales which reflect these smoothing properties becomes relatively straightforward.
- by finding an explicit functional relation between the X-ray transform (or rather, one of its normal operators) with distinguished differential operators. Since the latter oftentimes determine a scales of smoothness in their ambient spaces, the smoothing properties of the operator of interest naturally follow.

The next three examples include a compact manifold with boundary, a complete manifold, and a manifold with boundary. Each of these cases brings its own set of peculiarities.

Exercise 1 (On Fourier series)**.** *Given $f \in L^1(\mathbb{S}^1)$, we may define the sequence of Fourier coefficients of f, $\{c_n[f] := \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(\theta)\ d\theta\}_{n\in\mathbb{Z}}$, a bounded, doubly infinite sequence (in $\ell^\infty(\mathbb{Z})$). For fixed n, we denote*

$$S_n[f](\theta) := \sum_{k=-n}^{n} c_k[f] e^{ik\theta} \in P_n,$$

where we denote P_n the set of trigonometric polynomials of degree at most n, of the form $p(\theta) = \sum_{k=-n}^n c_k e^{ik\theta}$ for some complex numbers c_k.

On $[0, 2\pi]$, let us define the Hilbert scale

$$H^k(\mathbb{S}^1) = \left\{ f \in L^2(\mathbb{S}^1),\ \|f\|_k^2 := \sum_{j=0}^{k} \int_0^{2\pi} |f^{(j)}(\theta)|^2 < \infty \right\}, \tag{6}$$

with $H^0 = L^2$, equipped with the Hermitian inner product $(f, g) = \int_0^{2\pi} f(\theta)\bar{g}(\theta)\ d\theta$. On $\mathbb{Z}$, define the Hilbert scale

$$h^k(\mathbb{Z}) = \left\{ \mathbf{u} \in \mathbb{C}^{\mathbb{Z}},\ \|\mathbf{u}\|_k^2 = \sum_{j\in\mathbb{Z}} (1+j^2)^k |u_j|^2 < \infty \right\}. \tag{7}$$

1. *Check that the functions $\{\mathbf{e}_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}\}_{n\in\mathbb{Z}}$ is an orthonormal system in $L^2(\mathbb{S}^1)$ and that $S_n[f] = \sum_{k=-n}^n (f, \mathbf{e}_k)\mathbf{e}_k$. In the sequel, we will assume that $\{\mathbf{e}_n\}_{n\in\mathbb{Z}}$ is a* complete[6] *orthonormal set in $L^2(\mathbb{S}^1)$.*

2. *Show that $S_n[f]$ is the minimizer of functional*

$$F(p) = \int_0^{2\pi} |f(\theta) - p(\theta)|^2\ d\theta, \qquad p \in P_n.$$

   Thus, $S_n[f]$ is the best L^2-approximant of f among all trigonometric polynomials of degree n.

3. *Show that for every n, $S_n[f]$ and $f - S_n[f]$ are L^2-orthogonal and that*

$$\begin{aligned} \|f\|_{L^2}^2 &= \|S_n[f]\|_{L^2}^2 + \|f - S_n[f]\|_{L^2}^2 \\ &= 2\pi \sum_{k=-n}^{n} |c_k[f]|^2 + \|f - S_n[f]\|_{L^2}^2, \qquad n \ge 0. \end{aligned}$$

4. *Deduce that $S_n[f]$ converges to f in $L^2(\mathbb{S}^1)$ and that*

$$\|f\|_{L^2}^2 = 2\pi \sum_{k\in\mathbb{Z}} |c_k[f]|^2 \qquad (\textit{Parseval}). \tag{8}$$

   What does (8) say about the Fourier series map $L^2(\mathbb{S}^1) \ni f \mapsto \{c_k[f]\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$?

[6] This means: if $f \in L^2(\mathbb{S}^1)$ satisfies $(f, \mathbf{e}_n) = 0$ for all n, then $f = 0$. A proof can be found in [HN01, Ch. 7], showing that trigonometric polynomials are dense in $C(\mathbb{S}^1)$ using approximations of identity and the density of $C(\mathbb{S}^1)$ in $L^2(\mathbb{S}^1)$.

5. *Suppose $f \in C^1$. Show that $c_n[\frac{1}{i}\frac{d}{d\theta} f] = n\ c_n[f]$ for all $n \in \mathbb{Z}$.*

6. *Given a polynomial $q = \sum_{j=0}^{\ell} q_j x^j$, denote $q(\frac{1}{i}\frac{d}{d\theta}) := \sum_{j=0}^{\ell} q_j \left(\frac{1}{i}\frac{d}{d\theta}\right)^j$.*

   Provided that $f \in C^\ell$, show that $c_n[q(\frac{1}{i}\frac{d}{d\theta})f] = q(n)c_n[f]$. Combine this with (8) to deduce that $f \in H^\ell(\mathbb{S}^1)$, then its coefficients belong to $h^\ell(\mathbb{Z})$.

7. *Prove the converse: if a sequence belongs to $h^\ell(\mathbb{Z})$, then the Fourier series construct a function that is $H^\ell(\mathbb{S}^1)$.*

Exercise 2. *For $k \geq 0$, define the scale of Hilbert spaces*

$$h^k := \left\{ \mathbf{u} = \{u_n\}_{n\geq 0} \in \mathbb{C}^{\mathbb{N}},\ \|\mathbf{u}\|_k^2 := \sum_{n\geq 0} n^{2k}|u_n|^2 < \infty \right\}. \tag{9}$$

and denote $\ell^2 := h^0$. Consider a sequence of complex numbers $\{\lambda_n\}_{n\geq 0}$, and consider the 'diagonal' operator

$$A\colon \ell^2 \to \ell^2, \qquad A(\{u_n\}_n) = \{\lambda_n u_n\}_n.$$

1. *Under what condition is the operator A bounded (=continuous) ?*

2. *Suppose $\lambda_n \neq 0$ for all n. Is A invertible ? Is $A^{-1}\colon \ell^2 \to \ell^2$ always continuous ?*

3. *Fix $p \geq 0$ and suppose $\lambda_n = \frac{1}{n^p}$. With A defined as above, what is the order of ill-posedness of A in the scale h^k defined in (9) ?*

4. *Same question with the sequence $\lambda_n = e^{-n}$.*

Exercise 3 (Bessel solution operator)**.** *Consider the operator $\mathcal{M}$ defined in (2).*

1. *Describe the operator $\mathcal{M}$ using Fourier series.*

2. *What is the degree of ill-posedness of $\mathcal{M}$ relative to the scale (6) ?*

3. *Show that if $H^\infty := \cap_{k\geq 0} H^k(\mathbb{S}^1)$ is equipped with its Fréchet topology given by the seminorms $\{\|\cdot\|_k\}_k$ in (6), the operator $\mathcal{M}\colon H^\infty \to H^\infty$ has a continuous inverse.*

Exercise 4 (Diffusion operator)**.** *Consider the operator $\mathcal{M}$ defined in (1).*

1. *Describe the operator $\mathcal{M}$ using Fourier series.*

2. *Show that, relative to the scale (6), $\mathcal{M}$ is severely ill-posed.*

3. *Describe the "canonical" scale of spaces (4).*

Exercise 5 (On moduli of stability)**.** *Let a* linear *forward operator* $\mathcal{M}\colon \mathcal{X} \to \mathcal{Y}$ *and* $\mathcal{X} = \cap_{k\ge 0}\mathcal{X}_k$, $\mathcal{Y} = \cap_{k\ge 0}\mathcal{Y}_k$ *with* $\{\mathcal{X}_k\}_k$, $\{\mathcal{Y}_k\}_k$ *Banach scales. Suppose that for every* k, $\mathcal{M}\colon \mathcal{X}_k \to \mathcal{Y}_k$ *is bounded. Assume moreover that* $\{\mathcal{Y}_k\}_k$ *satisfies* interpolation inequalities[7] *of the form, if* $0 \le k \le \ell$

$$\|g\|_{\mathcal{Y}_{\lambda k+(1-\lambda)\ell}} \le C\|g\|_{\mathcal{Y}_k}^{\lambda}\|g\|_{\mathcal{Y}_\ell}^{1-\lambda}, \qquad \lambda \in [0,1], \quad g \in \mathcal{Y}_\ell, \tag{10}$$

for some constant $C(k,\ell,\lambda)$*. Suppose that the operator* $\mathcal{M}$ *is ill-posed of order* $\alpha > 0$ *in the sense that*

$$\|x - x'\|_{\mathcal{X}_k} \le C\|\mathcal{M}(x) - \mathcal{M}(x')\|_{\mathcal{Y}_{k+\alpha}}, \quad k \ge 0. \tag{11}$$

Fixing k*, the question is whether we can obtain an estimate of* $\|x - x'\|_{\mathcal{X}_k}$ *in terms of a LOWER-regularity norm of* $\mathcal{M}(x - x')$ *than* $\mathcal{Y}_{k+\alpha}$*, ideally* $\mathcal{Y}_k$ *(as this might be the space where the noise lives). We show that this is possible if we add the* prior *assumption that the unknown has "high regularity", namely* $x, x' \in \mathcal{X}_\beta$ *for some* $\beta > \alpha + k$.

1. *Assuming the uniform bound* $\|x\|_{\mathcal{X}_\beta}, \|x'\|_{\mathcal{X}_\beta} \le C$ *for some* $\beta > \alpha + k$*, use (10) for appropriate* (λ, k, ℓ)*, (11) and the boundedness of* $\mathcal{M}$ *to show a Hölder estimate of the form*

$$\|x - x'\|_{\mathcal{X}_k} \le C\|\mathcal{M}(x) - \mathcal{M}(x')\|_{\mathcal{Y}_k}^{\theta},$$

   where $\theta \in [0,1]$ *depends on* α *and* β.

2. *How does* θ *behave as* β *increases ?*

3. *Use Hölder's inequality to show that the scale* $\{h^k\}_k$ *in (9) satisfies (10).*

Exercise 6.

1. *Let* $\{\lambda_n\}_n$ *a sequence of non-negative numbers decreasing to zero, and define the operator* $A\colon \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ *by* $(A\mathbf{u})_n = \lambda_n u_n$, $\mathbf{u} = \{u_n\}_n$.

   Prove that the operator A *is compact.*

2. *Deduce that the inclusion* $h^2 \to \ell^2$ *is compact.*

[7] Sobolev scales most often satisfy this.

2 Lecture 2 - The Funk transform on the two-sphere $\mathbb{S}^2$

2.1 Formulation

Our first example of integral-geometric operator is the Funk transform, initially studied in [Fun16]. In the 70's, Guillemin [Gui76] used this transform in order to construct Zoll[8] metrics on the sphere. This example is also treated in [MP11, Sec. 1.2]. See also the recent work [Kaz18].

For the Funk transform, we can derive the full SVD of the operator, and appropriate Hilbert scales where to describe the mapping properties of the operator.

Let $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3, x^2+y^2+z^2 = 1\}$ the Euclidean 2-sphere. Given $f \in C^\infty(\mathbb{S}^2)$ and $p \in \mathbb{S}^2$, we define

$$If(p) := \int_0^{2\pi} f(\gamma_p(t))\ dt,$$

where γ_p is the equator on the sphere thinking of p as the North pole (it moves around !), traversed counterclockwise when viewed from p. If $p = N = (0,0,1)$ (the actual North Pole),

$$\gamma_N(t) = (\cos t, \sin t, 0), \qquad t \in [0, 2\pi), \tag{12}$$

parametrized as a nuit-speed geodesic.

Any other great circle can be deduced from γ_N via a rotation, in particular the group

$$SO(3) = \{R \in M_3(\mathbb{R}),\ R^T R = id,\ \det R = 1\} \tag{13}$$

will play a special role in what follows via its action on functions on the sphere (or on $\mathbb{R}^3$ in the same way): for $f \in C^\infty(\mathbb{S}^2)$ and $R \in SO(3)$, define

$$R \cdot f(\mathrm{x}) := f(R^{-1}\mathbf{x}) = f(R^T\mathbf{x}), \qquad \mathbf{x} \in \mathbb{S}^2. \tag{14}$$

Exercise 7. *1. How to parameterize γ_p for any $p \in \mathbb{S}^2$?*

2. Give the parametric equation of γ_p for $p = (0,1,0)$.

3. Give the parametric equation of γ_p for $p = (1,1,1)/\sqrt{3}$.

Since γ_p depends smoothly on p and $f \in C^\infty(\mathbb{S}^2)$, then $If \in C^\infty(\mathbb{S}^2)$. Note that in this case, f and If can both be viewed as functions on the same domain $\mathbb{S}^2$ (this will never happen again in the integral geometric problems considered below). The question to investigate is thus:

study the problem of reconstructing $f \in C^\infty(\mathbb{S}^2)$ from $If \in C^\infty(\mathbb{S}^2)$.

Exercise 8. *Consider $f\colon \mathbb{S}^2 \to \mathbb{R}$ equal to 1 on a spherical cap of small aperture $\alpha \le \pi/8$ centered at the North pole (in spherical coordinates $(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$, this corresponds to the set $\{\theta \in [0,\alpha]\}$), and zero elsewhere.*

Describe the support of If, and the sets on which If is comstant.

[8] A Zoll manifold is a closed Riemannian manifold, all of whose geodesics are closed and of the same length. The round sphere is such an example.

Obvious kernel and cokernel. Let $A : \mathbb{S}^2 \to \mathbb{S}^2$ the antipodal map ($A(p) = -p$). A is a smooth involution and induces a direct sum decomposition

$$C^\infty(\mathbb{S}^2) = C^\infty_{even}(\mathbb{S}^2) \oplus C^\infty_{odd}(\mathbb{S}^2),$$

where a function f will be considered even if $f \circ A = f$, odd if $f \circ A = -f$. Then the following two observations can be made: first $If = 0$ whenever f is odd, and If is even for any f. It therefore makes sense that we refine our initial question to:

$$\text{study the problem of reconstructing } f \in C^\infty_{even}(\mathbb{S}^2) \text{ from } If \in C^\infty_{even}(\mathbb{S}^2).$$

In what follows, we will show that I is an automorphism of $C^\infty_{even}(\mathbb{S}^2)$ and that, upon defining an appropriate Hilbert scale $H^s_{even}(\mathbb{S}^2)$, I is an isomorphism of H^s_{even} onto $H^{s+\frac{1}{2}}_{even}$. This will be based on understanding its eigendecomposition by means of spaces of spherical harmonics.

2.2 Spherical Harmonics

In order to define a scale of spaces on $\mathbb{S}^2$, one natural idea is to define, for $k = 2\ell$ even, $H^{2\ell}(\mathbb{S}^2)$ as the closure of $C^\infty(\mathbb{S}^2)$ for the topology defined by the norm $\|f\|_{H^{2\ell}} = \|(-\Delta_{\mathbb{S}^2} + 1)^\ell f\|_{L^2}$. Here $\Delta_{\mathbb{S}^2}$ denotes the Laplace-Beltrami operator[9], then $-\Delta_{\mathbb{S}^2}$ is non-negative, essentially self-adjoint operator and the $+1$ makes it injective. It is an example of an *elliptic* operator, which means that, in spite of the fact that this is a single differential operator, if Δf is in L^2, then so will be any other second derivative of f.

When this is done, we realize that we only have spaces of even order, though we would like to define H^s for all integer s, or even for all $s \ge 0$ (and even all $s \in \mathbb{R}$). A process called *interpolation* allows to do this, though another clear picture emerges if we know the spectral decomposition of $\Delta_{\mathbb{S}^2}$. Note that general functional-analytic arguments show that $\Delta_{\mathbb{S}^2}$ has a complete, discrete eigensystem in $L^2(\mathbb{S}^2)$ and that its spectrum tends to ∞[10]. We now undertake this task, largely following [Fol95, Ch. 2.H].

The story of spherical harmonics sums up to this[11]: define $\mathcal{H}_k$ the space of polynomials on $\mathbb{R}^3$, harmonic[12] and homogeneous of degree k, and let $H_k = \{P|_{\mathbb{S}^2} : P \in \mathcal{H}_k\}$. The latter is called *spherical harmonics of degree k*.

Theorem 1. *(Spherical harmonics) (1) For every $k \in \mathbb{N}_0$, $H_k = \ker(-\Delta_{\mathbb{S}^2} - k(k+1)Id)$ and $\dim H_k = 2k+1$. Moreover, H_k is an irreducible $SO(3)$-representation.*

(2) We have the direct orthogonal sum

$$L^2(\mathbb{S}^2) = \bigoplus_{k\ge 0} H_k, \tag{15}$$

[9] In the coordinates $(\theta, \varphi) \mapsto (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, $\Delta_{\mathbb{S}^2} f = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \frac{\partial f}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$

[10] Sketch of proof: by Riesz-representation theorem, for $f \in L^2$, the problem $-\Delta_{\mathbb{S}^2} u + u = -f$ admits a unique solution in H^1, this defines $(-\Delta_{\mathbb{S}^2} + 1)^{-1} : L^2(\mathbb{S}^2) \to H^1(\mathbb{S}^2)$ as a bounded operator. Since the inclusion $H^1 \to L^2$ is compact, then $(-\Delta_{\mathbb{S}^2} + 1)^{-1}$ is a compact, operator, moreover, injective and self-adjoint. By the spectral theorem for compact, self-adjoint operators ([Fol95, Th. 0.44]), there exists a complete orthonormal set ϕ_n of $L^2(\mathbb{S}^2)$ along with a decreasing sequence $\lambda_n \to 0$ such that $(-\Delta_{\mathbb{S}^2} + 1)^{-1}\phi_n = \lambda_n \phi_n$. Then $(\phi_n, \lambda_n^{-1} - 1)$ is an eigensystem for $\Delta_{\mathbb{S}^2}$.

[11] it works with minor modifications in all dimensions ≥ 2, see [Fol95, Ch. 2.H]

[12] in the sense that $\Delta_{\mathbb{R}^3} P = 0$, where $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$

in the sense that every $f \in L^2(\mathbb{S})$ admits a unique orthogonal and L^2-convergent decomposition $f = \sum_{k\ge 0} f_k$ with $f_k \in H_k$.

Proof of Theorem 1. First let $\mathcal{P}_k$ the space of polynomials on $\mathbb{R}^3$, homogeneous of degree k, and write $r^2 := x^2 + y^2 + z^2$. One may show as exercise that $\dim \mathcal{P}_k = (k+1)(k+2)/2$. In addition, the action (14) preserves $\mathcal{P}_k$. This is due to the fact that the Laplacian commutes with this action (see Exercise 11), and that this action preserves homogeneity.

One must then understand the orthocomplement of $\mathcal{H}_k$ in $\mathcal{P}_k$. To this effect, [Fol95, Prop. 2.49] states that for $k \ge 2$,

$$\mathcal{P}_k = \mathcal{H}_k \oplus r^2\mathcal{P}_{k-2}, \qquad r^2\mathcal{P}_{k-2} := \{r^2 P : P \in \mathcal{P}_{k-2}\}, \tag{16}$$

whose proof is given in Ex. 10. As a result,

$$\dim \mathcal{H}_k = \dim \mathcal{P}_k - \dim \mathcal{P}_{k-2} = 2k+1,$$

and by induction,

$$\mathcal{P}_k = \mathcal{H}_k \oplus r^2\mathcal{H}_{k-2} \oplus r^4\mathcal{H}_{k-4} \dots \tag{17}$$

Also recall the expression of the Laplacian in spherical coordinates:

$$\Delta_{\mathbb{R}^3} = \frac{1}{r^2}\Delta_{\mathbb{S}^2} + \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}.$$

Now, if $f \in \mathcal{H}_k$, then one may write $f = r^k \bar{f}$ where $\bar{f} = f|_{\mathbb{S}^2}$, then applying the equation above and evaluating at $r = 1$ yields the relation

$$\Delta_{\mathbb{R}^3} f = 0 = \Delta_{\mathbb{S}^2}\bar{f} + k(k+1)\bar{f}.$$

Hence H_k consists is the eigenspace of $\Delta_{\mathbb{S}^2}$ associated with eigenvalue $-k(k+1)$. As $\Delta_{\mathbb{S}^2}$ is self-adjoint, this implies the $L^2(\mathbb{S}^2)$-orthogonality $H_k \perp H_\ell$ for $k \ne \ell$. A proof of the irreducibility can be found in [Fol95, Exercise 8, Ch 2.H].

The proof of (15) is based on the Weierstrass approximation theorem, together with (17): in a nutshell, a function in $L^2(\mathbb{S}^2)$ can be approximated by functions in $C(\mathbb{S}^2)$, which in turn can be approximated by restrictions to $\mathbb{S}^2$ of polynomials, which by (17) decompose as finite sums of spherical harmonics. □

Exercise 9. *Show that* $\dim \mathcal{P}_k = (k+1)(k+2)/2$.

Exercise 10. *This problem guides you through the proof of (16). Here and below, denote* $\mathrm{x} = (x, y, z)$ *and for a tri-index* $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$*, we denote* $|\alpha| = \alpha_1+\alpha_2+\alpha_3$*,* $\alpha! = \alpha_1!\alpha_2!\alpha_3!$*,* $\mathrm{x}^\alpha = x^{\alpha_1}y^{\alpha_2}z^{\alpha_3}$ *and* $\partial^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_z^{\alpha_3}$*. Thus, a general element of* $\mathcal{P}_k$ *takes the form* $P = \sum_{|\alpha|=k} a_\alpha \mathrm{x}^\alpha$ *for some complex numbers* $\{a_\alpha\}_\alpha$*, where the sum runs over all tri-indices of length* k*, and for such a* P*, we define the differential operator* $P(\partial) := \sum_{|\alpha|=k} a_\alpha \partial^\alpha$.

1. *On $\mathcal{P}_k$, we define the inner product*

$$\left(P = \sum_{|\alpha|=k} a_\alpha \mathrm{x}^\alpha, Q = \sum_{|\beta|=k} b_\beta \mathrm{x}^\beta\right) \mapsto \sum_{|\alpha|=k} \alpha! a_\alpha \overline{b_\alpha}.$$

Show that such an inner product can be obtained by computing the quantity $\{P, Q\} := P(\partial)\overline{Q}$. [Hint: show that $\{\mathrm{x}^\alpha, \mathrm{x}^\beta\} = \alpha!$ if $\alpha = \beta$, 0 otherwise.]

2. *Show that for $P \in \mathcal{P}_{k-2}$ and $Q \in \mathcal{P}_k$,*

$$\{r^2 P, Q\} = \{P, \Delta_{\mathbb{R}^3} Q\}.$$

3. *Conclude (16).*

Exercise 11 (Rotation-invariance of the Laplacian)**.** *Let $\Delta_{\mathbb{R}^3} = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$, $f \in C^\infty(\mathbb{R}^3)$ and $R \in SO(3)$. Define $h(\mathbf{x}) := f(R^T\mathbf{x})$ (or, equivalently, $f(R^{-1}\mathbf{x})$).*

Show that $\Delta_{\mathbb{R}^3} h(\mathbf{x}) = (\Delta_{\mathbb{R}^3} f)(R^T\mathbf{x})$. In terms of the action (14), this means that $\Delta_{\mathbb{R}^3}(R\cdot f) = R\cdot(\Delta_{\mathbb{R}^3} f)$ for all $f \in C^\infty(\mathbb{R}^3)$ and $R \in SO(3)$.

2.3 Eigendecomposition of the Funk transform

The group $SO(3)$ acts on $C^\infty(\mathbb{S}^2)$ by rotations: $g\cdot f(p) = f(g^{-1}p)$. The key observation is that I commutes[13] with this action in the sense that

$$I(g\cdot f) = g\cdot If, \qquad g \in SO(3), \qquad f \in C^\infty(\mathbb{S}^2).$$

By *Schur's lemma*[14], this implies two things at the level of all the irreducible $SO(3)$-representations H_k: for all $k \geq 0$,

- $I(H_k) \subseteq H_k$
- $I|_{H_k} = \lambda_k Id$ for some constant λ_k.

If k is odd, since $H_k \subset C^\infty_{odd}(\mathbb{S}^2)$, we already know that $\lambda_k = 0$. For k even, $k = 2\ell$ for some $\ell \geq 0$, it suffices to find a "good" function in $f \in H_{2\ell}$ and a point p where $f(p) \neq 0$, then $\lambda_{2\ell}$ will be given by $c_{2\ell} = \frac{If(p)}{f(p)}$. A good choice is as follows: one may check that the "sectoral" harmonic function $f(x,y,z) = (x+iy)^{2\ell}$, restricted to $\mathbb{S}^2$, belongs to $H_{2\ell}$. Now choose $p = (1,0,0)$, with $\gamma_p(t) = (0, \cos t, \sin t)$ to deduce

$$\lambda_{2\ell} = \frac{1}{f(1,0,0)} \int_0^{2\pi} f(0, \cos t, \sin t)\ dt = (-1)^\ell \int_0^{2\pi} (\cos t)^{2\ell}\ dt.$$

[13]In representation theory language, I is equivariant w.r.t the $SO(3)$ action, or I is $SO(3)$-linear.

[14]Schur's lemma: Suppose V, W are complex vector spaces with two irreducible G-representations ρ_V, ρ_W. (1) If V and W are not isomorphic, then there exists no non-trivial G-linear map between them. (2) If $V = W$ and $\rho_V = \rho_W$, then the only G-linear maps between V and W are the constant multiples of the identity.

The above is a *Wallis integral*, and one may deduce the final expression

$$\lambda_{2\ell} = (-1)^\ell 2\pi \frac{(2\ell)!}{2^{2\ell}(\ell!)^2}, \qquad \ell \in \mathbb{N}_0. \tag{18}$$

As a conclusion, the eigenvalue decomposition of $I\colon L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ is given by:

$$\ker I = \bigoplus_{\ell \ge 0} H_{2\ell+1}, \quad \ker(I - \lambda_{2\ell}) = H_{2\ell}, \qquad \ell \ge 0,$$

or at the level of the spectral decomposition:

$$If = \sum_{k\ge 0} \lambda_k f_k, \qquad f = \sum_{k\ge 0} f_k, \qquad f_k \in H_k.$$

Using *Stirling's formula*[15], we arrive at the conclusion that

$$|\lambda_{2\ell}| \sim \sqrt{8\pi}(2\ell)^{-1/2}, \qquad \ell \to \infty. \tag{19}$$

We now explain how to exploit this to formulate mapping properties of the Funk transform in a sharp way.

2.4 Mapping properties

To formulate mapping properties, we need two things: (1) to translate smoothness on $\mathbb{S}^2$ into rate of decay of the spherical harmonic decomposition and (2) to use the asymptotics of $\lambda_{2\ell}$ for large ℓ given in (19).

For $s \ge 0$, one may define $H^s(\mathbb{S}^2)$ to be the closure of $C^\infty(\mathbb{S}^2)$ for the norm

$$\|f\|_{H^s}^2 := \sum_{k\ge 0} (1 + k(k+1))^s \|f_k\|_{L^2}^2, \qquad f = \sum_{k\ge 0} f_k, \quad f_k \in H_k. \tag{20}$$

For $s = 2\ell$ an even integer, this amounts to the norm $\|(-\Delta+1)^\ell f\|_{L^2}$ and hence it indeed encodes that, when this quantity is finite, the function f has 2ℓ square-integrable derivatives. Allowing s to be any non-negative reals achieves a few things: it gives meaning to "having derivatives of fractional order"; it hints at the fact that for any $s \ge 0$, the operator

$$f = \sum_{k\ge 0} f_k \mapsto \sum_{k \ge 0} (1 + k(k+1))^{s/2} f_k$$

is one way to define the operator $(-\Delta+1)^{s/2}$. By construction, this operator is $H^s \to L^2$ bounded, a process which embodies the action of taking "s derivatives" (in an "isotropic" way).

[15] $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ as $n \to \infty$

Remark 1. *Given s fixed, although the construction (20) has the special property that it can be related exactly to the operator $(-\Delta + 1)$, one may notice that for* any *sequence d_k such that (i) $d_k \neq 0$ for all k and (ii) there exists two positive constants such that $C_1 \leq |k^s d_k| \leq C_2$, one may define the norm*

$$\|f\|^2 := \sum_{k\geq 0} d_k^2 \|f_k\|_{L^2}^2, \qquad f = \sum_{k\geq 0} f_k, \quad f_k \in H_k,$$

and the closure of $C^\infty(\mathbb{S}^2)$ with respect to that norm would give a Hilbert space whose topology is the same as $H^s(\mathbb{S}^2)$.

Just like $C^\infty = C^\infty_{even} \oplus C^\infty_{odd}$, we can split these Sobolev spaces into even and odd functions

$$H^s(\mathbb{S}^2) = H^s_{even}(\mathbb{S}^2) \oplus H^s_{odd}(\mathbb{S}^2).$$

On to the smoothing properties of I, the main crux is to understand the polynomial behavior of $\lambda_{2\ell}$ as $\ell \to \infty$. By (19), there exists two positive constants C_1, C_2 such that

$$C_1(1 + (2\ell)^{1/2}) \leq |\lambda_{2\ell}| \leq C_2(1 + (2\ell)^{1/2}), \qquad \ell \geq 0. \tag{21}$$

Out of this, we can deduce the two-sided estimates

$$C_1 \|f\|_{H^s_{even}} \leq \|If\|_{H^{s+\frac{1}{2}}_{even}} \leq C_2 \|f\|_{H^s_{even}}, \tag{22}$$

a precise description of the fact that the operator I is smoothing of order $\frac{1}{2}$. Inequalities (22) can first be proved for $f \in C^\infty_{even}(\mathbb{S}^2)$, then extended by density to the space $H^s_{even}(\mathbb{S}^2)$.

Exercise 12. *Work out the details of (21) and (22).*

3 Lecture 3 - The Radon transform on the plane $\mathbb{R}^2$

3.1 Preliminaries

Recall that the Schwartz space of "smooth functions with rapid decay" (e.g. gaussians, compactly supported smooth functions) is the Fréchet space $\mathscr{S}(\mathbb{R}^2)$ whose topology is defined by the countable family of seminorms

$$\|f\|_{\alpha,\beta} \le \sup_{\mathbb{R}^2} |\mathbf{x}^\alpha \partial^\beta f(\mathbf{x})|, \qquad \alpha, \beta \in \mathbb{N}_0^2.$$

In other words, $\{f_n\}_n$ converges to f in $\mathscr{S}$ if for every bi-indices α, β, $\|f_n - f\|_{\alpha,\beta} \to 0$ as $n \to \infty$. The Fourier transform

$$\mathcal{F}\colon f \mapsto \hat{f}(\xi) = \int_{\mathbb{R}^2} f(\mathbf{x}) e^{-i\mathbf{x}\cdot\xi}\ d\xi, \qquad \xi \in \mathbb{R}^2, \tag{23}$$

is a well-defined and continuous map from $\mathscr{S}(\mathbb{R}^2)$ to itself, thanks to the following result:

Exercise 13. *Show that for any multi-indices α, β and $f \in \mathscr{S}(\mathbb{R}^2)$,*

$$\mathcal{F}[\mathbf{x}^\alpha \partial_{\mathbf{x}}^\beta f](\xi) = i^{|\alpha|+|\beta|} \partial_\xi^\alpha (\xi^\beta \hat{f})(\xi).$$

Deduce that $\mathcal{F}\colon \mathscr{S} \to \mathscr{S}$ is continuous.

The inversion of $\mathcal{F}$ is given by

$$f(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(\xi) e^{i\mathbf{x}\cdot\xi}\ d\xi, \qquad \mathbf{x} \in \mathbb{R}^2.$$

The space of *tempered distributions*, $\mathscr{S}'(\mathbb{R}^2)$, is the dual space to $\mathscr{S}(\mathbb{R}^2)$, i.e. the space of linear, continuous forms on $\mathscr{S}(\mathbb{R}^2)$. In other words, u belongs to $\mathscr{S}'$ if for every $\{f_n\}_n$ converging to f in $\mathscr{S}$, the pairings $\langle u, f_n\rangle$ converge to $\langle u, f\rangle$. One way to characterize them is: u belongs to $\mathscr{S}'$ if and only if there exists $k \in \mathbb{N}_0$ and a constant C such that

$$|u(f)| \le C \sum_{|\alpha|,|\beta| \le k} \|f\|_{\alpha,\beta}. \tag{24}$$

A function $f \in \mathscr{S}(\mathbb{R}^2)$ gives rise to a tempered distribution $u_f \in \mathscr{S}'(\mathbb{R}^2)$ via the action

$$u_f\colon g \mapsto \langle f, g\rangle := \int_{\mathbb{R}^2} f\ g\ d\mathbf{x}$$

and, together with the identity

$$\int_{\mathbb{R}^2} \hat{f}(\xi)\ g(\xi)\ d\xi = \int_{\mathbb{R}^2} f(\mathbf{x}) \hat{g}(\mathbf{x})\ d\mathbf{x},$$

this motivates an extension of $\mathcal{F}$ to $\mathscr{S}'(\mathbb{R}^2)$ through the definition

$$\langle \hat{u}, f\rangle := \langle u, \hat{f}\rangle, \qquad f \in \mathscr{S}(\mathbb{R}^2). \tag{25}$$

The Fourier transform so obtained is a continuous map $\mathcal{F}\colon \mathscr{S}'(\mathbb{R}^2) \to \mathscr{S}'(\mathbb{R}^2)$ which allows to make sense of the Fourier transform for rather badly behaved objects[16].

[16] think: functions with low-regularity, no-decay at infinity, diracs, etc...

3.2 Definition and basic mapping observations

Given $f \in C_c(\mathbb{R}^2)$ or $f \in \mathscr{S}(\mathbb{R}^2)$, we define

$$Rf(s,\theta) = \int_{\mathbb{R}} f(s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp)\ dt, \qquad (s,\theta) \in \mathcal{Z} := \mathbb{R}\times\mathbb{S}^1, \tag{26}$$

where $\boldsymbol{\theta} = \binom{\cos\theta}{\sin\theta}$ and $\boldsymbol{\theta}^\perp = \binom{-\sin\theta}{\cos\theta}$. Let us equip $\mathcal{Z}$ with area element $ds\ d\theta$ and define $L^2(\mathcal{Z})$ with respect to that measure. For the same reason that the Funk transform automatically produces even functions with respect to the antipodal map (integration of a function along a curve is insensitive to the orientation of that curve), the Radon transform of a function automatically satisfies the symmetry:

$$Rf(s,\theta) = Rf(-s,\theta+\pi), \qquad (s,\theta)\in\mathcal{Z}, \qquad f \in \mathscr{S}(\mathbb{R}^2). \tag{27}$$

One can define a Schwartz space on $\mathscr{S}(\mathcal{Z})$ through the seminorms

$$\|g\|_{k,\ell,m} = \sup_{\mathcal{Z}} |s^k \partial_s^\ell \partial_\theta^m g(s,\theta)|, \quad k,\ell,m \in \mathbb{N}_0, \tag{28}$$

and show that $R\colon \mathscr{S}(\mathbb{R}^2) \to \mathscr{S}(\mathcal{Z})$ is continuous.

Exercise 14. *Show that $R\colon \mathscr{S}(\mathbb{R}^2)\to\mathscr{S}(\mathcal{Z})$ is continuous. This can be done by showing that for every $p \in \mathbb{N}_0$, there exists $q\in\mathbb{N}_0$ and a constant $C_p > 0$ such that for all $f \in \mathscr{S}(\mathbb{R}^2)$,*

$$\sum_{k,\ell,m\le q} \|Rf\|_{k,\ell,m} \le C \sum_{|\alpha|,|\beta|\le p} \|f\|_{\alpha,\beta}.$$

Similarly, $R\colon C_c^\infty(\mathbb{R}^2) \to C_c^\infty(\mathcal{Z})$ is continuous[17]. Observe in particular that if f is supported in $\{|x| < R\}$, then Rf is supported in the truncated cylinder $\{|s| < R\}$.

Exercise 15.

1. *Compute the Radon transform of the characteristic function of the ball of radius $\epsilon > 0$ and center $s_0 e^{i\beta}$ with $\rho > 0$.*
2. *Let* $f(\mathrm{x}) = \begin{cases} (1-|\mathrm{x}|^2)^{-1/2} & |\mathrm{x}| < 1, \\ 0 & |\mathrm{x}| \ge 1. \end{cases}$ *Compute $Rf(s,\theta)$ for $|s| \le 1$.*

The necessity of moment conditions: lack of surjectivity The operator $R\colon \mathscr{S}(\mathbb{R}^2) \to \mathscr{S}(\mathcal{Z})$ is not surjective. To see this, we have the following:

Lemma 2. *If g is in the range of $R\colon \mathscr{S}(\mathbb{R}^2)\to\mathscr{S}(\mathcal{Z})$, then condition (27) is satisfied, and for every $k\in\mathbb{N}_0$, $\int_{\mathbb{R}} g(s,\theta)s^k\ ds$ is a homogeneous polynomial of degree k in $\cos\theta, \sin\theta$.*

[17] Note that spaces of smooth functions with compact support are slightly more general than Fréchet.

Proof. Suppose $g = Rf$ for some $f \in \mathscr{S}(\mathbb{R}^2)$ and let $k \ge 0$. We compute

$$\begin{aligned}\int_{\mathbb{R}} s^k Rf(s,\theta)\ ds &= \int_{\mathbb{R}^2} s^k f(s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp)\ dt\ ds \\ &= \int_{\mathbb{R}^2} (\mathrm{x}\cdot\boldsymbol{\theta})^k f(\mathrm{x})\ dx \qquad (\text{setting } \mathrm{x}(s,t) = s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp) \\ &= \sum_{j=0}^k \binom{k}{j} \cos^j\theta \sin^{k-j}\theta \int_{\mathbb{R}^2} f(\mathrm{x}) x^j y^{k-j}\ d\mathrm{x},\end{aligned}$$

hence the result. $\square$

For example, $g(s,\theta) = e^{-s^2} e^{i\theta}$ cannot be in the range of R, since $\int_{\mathbb{R}} g(s,\theta)\ ds = \sqrt{2\pi} e^{i\theta}$ is not a polynomial of degree zero in $\cos\theta, \sin\theta$. More generally, one may cook up enough linearly independent examples to show that the $\mathscr{S}(\mathcal{Z})\backslash R(\mathscr{S}(\mathbb{R}^2))$ is infinite-dimensional.

It can be shown that the converse also holds, namely that if these conditions are satisfied, then $g = Rf$ for some f. In fact, such an f can be uniquely constructed from the moments $\int_{\mathbb{R}^2} f(\mathrm{x}) x^j y^{k-j}\ d\mathrm{x}$ appearing above.

As a consequeence of the previous calculation, we can cook up a proof of injectivity of R on smooth (or merely continuous), compactly supported functions.

Theorem 3 ($C_c(\mathbb{R}^2)$-injectivity of R)**.** *Suppose $f \in C_c(\mathbb{R}^2)$ is such that $Rf = 0$. Then $f \equiv 0$.*

Proof. Let K be a compact set containing the support of f. By the calculations of lemma 2, the condition $Rf = 0$ implies that $\int_K f(\mathbf{x}) p(\mathbf{x})\ d\mathbf{x} = 0$ for all polynomials. Since K is compact, by Weiestrass approximation (density of polynomials in the uniform norm in $C(K)$, and a fortiori in the $L^2(K)$-norm), this forces $f = 0$. $\square$

Exercise 16. *Let $f \in \mathcal{S}(\mathbb{R}^2)$.*

1. *Show that $R[\partial_x f] = \cos\theta \frac{\partial}{\partial s} Rf$ and $R[\partial_y f] = \sin\theta \frac{\partial}{\partial s} Rf$.*
2. *Show that $\frac{\partial^2}{\partial s^2} Rf(s,\theta) = R[\Delta f](s,\theta)$.* $\qquad \Delta = \partial_x^2 + \partial_y^2$.
3. *For $(\alpha, \mathbf{a}) \in \mathbb{S}^1 \times \mathbb{R}^2$, define the action of the Euclidean group*

$$(\alpha, \mathbf{a}) \cdot f(\mathrm{x}) := f(R(\alpha)\mathrm{x} + \mathbf{a}), \qquad R(\alpha) := \left[\begin{array}{cc} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{array}\right], \qquad f \in \mathscr{S}(\mathbb{R}^2).$$

Find a relation between $R[(\alpha, \mathbf{a}) \cdot f]$ and Rf.

3.3 The $L^2 - L^2$ adjoint R^t

By direct calculation, one may compute the $L^2(\mathbb{R}^2) \to L^2(\mathcal{Z})$ adjoint R^t given by

$$R^t g(\mathrm{x}) = \int_{\mathbb{S}^1} g(\mathrm{x}\cdot\boldsymbol{\theta}, \theta)\ d\theta, \qquad \mathrm{x} \in \mathbb{R}^2. \tag{29}$$

Notice the duality of geometries: the Radon transform integrates a function over all points through a line, while its transpose integrates over all lines through a point.

3.4 The normal operator $R^t R$ as a convolution operator

Combining both definitions of R and R^t, one may obtain directly that

$$R^t R f(\mathrm{x}) = 2\int_{\mathbb{R}^2} f(\mathrm{y}) \frac{1}{|\mathrm{x}-\mathrm{y}|}\ d\mathrm{y} = \left(\frac{2}{|\cdot|} \star f\right)(\mathrm{x}).$$

Since $R^t R$ is a convolution operator, is becomes diagonalized through the Fourier transform, more specifically

$$\widehat{R^t R f}(\xi) = 2\hat{h}(\xi)\hat{f}(\xi), \qquad h(\mathrm{x}) := \frac{1}{|\mathrm{x}|}.$$

The question is: what sense does the Fourier transform of h make, and how to compute it ?

Indeed, h is not in L^1, nor in L^2... On the other hand, it makes sense as a tempered distribution, since for $f \in \mathscr{S}(\mathbb{R}^2)$,

$$\left|\int_{\mathbb{R}^2} \frac{1}{|\mathrm{x}|} f(\mathrm{x})\ d\mathrm{x}\right| \le \sup_{\mathrm{x}} |f(\mathrm{x})(1+|\mathrm{x}|^2)| \int_{\mathbb{R}^2} \frac{1}{|\mathrm{x}|(1+|\mathrm{x}|^2)}\ d\mathrm{x} \le C \sup_{\mathrm{x}} |f(\mathrm{x})(1+|\mathrm{x}|^2)|.$$

This justifies that its Fourier transform makes sense as a tempered distribution.

Lemma 4. *With $h(\mathbf{x}) = \frac{1}{|\mathbf{x}|}$, we have $\hat{h}(\xi) = \frac{2\pi}{|\xi|}$ in the sense of tempered distributions.*

Proof. To compute it, we use the following trick: the function h is the $\mathscr{S}'$-limit of $h_\epsilon(\mathrm{x}) = \frac{e^{-\varepsilon|\mathrm{x}|}}{|\mathrm{x}|}$ in the sense that for every $f \in \mathscr{S}$, $\langle h_\epsilon, f\rangle_{\mathscr{S}',\mathscr{S}} \to \langle h, f\rangle_{\mathscr{S}',\mathscr{S}}$ as $\epsilon \to 0$. Since the Fourier transform is $\mathscr{S}' \to \mathscr{S}'$-continuous, $\hat{h}$ is the $\mathscr{S}'$-limit of $\widehat{h_\epsilon}$, and since $h_\epsilon \in L^1(\mathbb{R}^2)$, we can compute its Fourier transform in the 'classical' sense:

$$\begin{aligned}\widehat{h_\epsilon}(\xi) = \int_{\mathbb{R}^2} \frac{e^{-\epsilon|\mathrm{x}|}}{|\mathrm{x}|} e^{-i\mathrm{x}\cdot\xi}\ d\mathrm{x} = \int_0^\infty \int_{\mathbb{S}^1} e^{-\epsilon\rho - i\rho|\xi|\cos\theta}\ d\theta\ d\rho &= \int_{\mathbb{S}^1} \frac{1}{\epsilon + i|\xi|\cos\theta}\ d\theta \\ &= \frac{1}{|\xi|}\int_{\mathbb{S}^1} \frac{1}{\frac{\epsilon}{|\xi|} + i\cos\theta}\ d\theta.\end{aligned}$$

Now by complex integration, one can show that for any $a > 0$,

$$\int_{\mathbb{S}^1} \frac{d\theta}{a + i\cos\theta} = \frac{2\pi}{\sqrt{1+a^2}}, \tag{30}$$

and hence,

$$\widehat{h_\epsilon}(\xi) = \frac{2\pi}{|\xi|} \frac{1}{\sqrt{1+\frac{\epsilon^2}{|\xi|^2}}},$$

whose $\mathscr{S}'$-limit can be proved to be $\hat{h}(\xi) = \frac{2\pi}{|\xi|}$. □

Exercise 17. *Prove (30).*

Conclusion. We then conclude that

$$\widehat{R^t R f}(\xi) = \frac{4\pi}{|\xi|}\hat{f}(\xi).$$

In particular, note that since $\widehat{-\Delta f}(\xi) = |\xi|^2 \hat{f}(\xi)$, then

$$\mathcal{F}((R^t R)^2(-\Delta) f)(\xi) = \frac{(4\pi)^2}{|\xi|^2}|\xi|^2\hat{f}(\xi) = (4\pi)^2\hat{f}(\xi).$$

In other words, $(R^tR)^2(-\Delta)f = (4\pi)^2 f$, a statement which one may think of as "the operator $\frac{1}{4\pi}R^tR$ is a negative squareroot of $(-\Delta)$".

What remains to clarify is: in what spaces does all of this work ?

3.5 Mapping properties on weighted L^2 spaces

Recall the definition, for $f \in \mathscr{S}(\mathbb{R}^2)$:

$$Rf(s,\theta) = \int_{\mathbb{R}} f(s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp)\ dt, \qquad (s,\theta) \in \mathbb{R}\times\mathbb{S}^1.$$

One may show that $R\colon \mathscr{S}(\mathbb{R}^2) \to \mathscr{S}(\mathcal{Z})$ is continuous. However, in spite of the fact that we have defined the "adjoint" of R relative to the $L^2(\mathbb{R}^2)$-$L^2(\mathcal{Z})$ pairing, let us first clarify the following point:

Lemma 5. *The Radon transform $R\colon L^2(\mathbb{R}^2) \to L^2(\mathcal{Z})$ is not bounded.*

Proof. Consider the function $f(\mathrm{x}) = \frac{1}{\langle \mathrm{x}\rangle^\beta}$ where we denote $\langle \mathrm{x}\rangle := (1+|\mathrm{x}|^2)^{1/2}$. We make the following claims:

- $f \in L^2$ iff $\beta > 1$: indeed,

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}^2} \frac{d\mathrm{x}}{(1+|\mathrm{x}|^2)^\beta} = 2\pi\int_0^\infty \frac{\rho\ d\rho}{(1+\rho^2)^\beta},$$

  and the latter is finite iff $2\beta - 1 > 1$.

- If $\beta > 1$, its Radon transform takes the form

$$Rf(s,\theta) = \int_{\mathbb{R}} \frac{1}{(1+s^2+t^2)^{\beta/2}}\ dt = \frac{1}{\langle s\rangle^{\beta-1}}\int_{\mathbb{R}}\frac{du}{(1+u^2)^{\beta/2}} = \frac{A_\beta}{\langle s\rangle^{\beta-1}},$$

  upon changing variable $t = u\sqrt{1+s^2}$. A_β is a fixed number, finite iff $\beta > 1$.

- For $\beta > 1$, $Rf \in L^2(\mathcal{Z})$ iff $\beta > \frac{3}{2}$: indeed,

$$\|Rf\|_{L^2(\mathcal{Z})}^2 = A_\beta^2 2\pi \int_{\mathbb{R}} \frac{ds}{\langle s\rangle^{2\beta-2}},$$

  and the integral on the right is finite iff $2\beta - 2 > 1$.

As a conclusion for any $1 < \beta \le 3/2$, the function $f(\mathrm{x}) = \langle \mathrm{x} \rangle^{-\beta}$ is in L^2 while Rf is not in L^2. $\square$

To reintroduce spaces in which the Radon transform becomes bounded, we need to consider weighted spaces where the weights behave polynomially at infinity. In particular, we define

$$L^2(\mathbb{R}^2, \langle \mathrm{x} \rangle^\alpha) := \left\{ f : \int_{\mathbb{R}^2} |f(\mathrm{x})|^2 \langle \mathrm{x} \rangle^\alpha \ d\mathrm{x} < \infty \right\},$$
$$L^2(\mathcal{Z}, \langle s \rangle^\alpha) := \left\{ g : \int_{\mathcal{Z}} |g(s,\theta)|^2 \langle s \rangle^\alpha \ ds \ d\theta < \infty \right\}.$$

With these definitions, we prove the following:

Theorem 6. *For any $\alpha > \frac{1}{2}$, the Radon transform is continuous in the following setting:*

$$R\colon L^2(\mathbb{R}^2, \langle \mathrm{x} \rangle^{2\alpha}) \to L^2(\mathcal{Z}, \langle s \rangle^{2\alpha-1}),$$

with operator norm no greater than $\sqrt{2\pi A_{2\alpha}}$, $A_{2\alpha} := \int_{\mathbb{R}} \frac{dt}{(1+t^2)^\alpha}$.

Proof. Suppose $\alpha > \frac{1}{2}$, then use Cauchy-Schwarz inequality to make appear

$$\begin{aligned} |Rf|(s,\theta)^2 &= \left| \int_{\mathbb{R}} f(s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp) \frac{(1+s^2+t^2)^{\alpha/2}}{(1+s^2+t^2)^{\alpha/2}} \ dt \right| \\ &\le \int_{\mathbb{R}} |f(s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp)|^2 \langle s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp \rangle^\alpha \ dt \cdot \int_{\mathbb{R}} \frac{dt}{(1+s^2+t^2)^\alpha}. \end{aligned}$$

Changing variable $t = \sqrt{1+s^2}u$, we arrive at

$$\int_{\mathbb{R}} \frac{dt}{(1+s^2+t^2)^\alpha} = \frac{A_{2\alpha}}{\langle s \rangle^{2\alpha-1}}, \qquad A_{2\alpha} = \int_{\mathbb{R}} \frac{dt}{(1+t^2)^\alpha} < \infty.$$

Multiplying through by $\langle s \rangle^{2\alpha-1}$, we arrive at

$$|Rf|(s,\theta)^2 \langle s \rangle^{2\alpha-1} \le A_{2\alpha} \int_{\mathbb{R}} |f(s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp)|^2 \langle s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp \rangle^\alpha \ dt.$$

Now integrate w.r.t. $ds \ d\theta$, change variable $\mathrm{x}(s,t) = s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp$ in the R.H.S. to arrive at the estimate

$$\|Rf\|^2_{L^2(\mathcal{Z}, \langle s \rangle^{2\alpha-1})} \le 2\pi A_{2\alpha} \|f\|^2_{L^2(\mathbb{R}^2, \langle \mathrm{x} \rangle^{2\alpha})}.$$

Hence the result. $\square$

Exercise 18. *Find an expression for A_β in terms of the Beta function*

$$B(x,y) := 2\int_0^{\pi/2} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1} \ d\theta, \qquad x > 0, \ y > 0.$$

Remark 2. *For any $\alpha > 1/2$, we can now compute the adjoint of $R\colon L^2(\mathbb{R}^2, \langle \mathbf{x} \rangle^{2\alpha}) \to L^2(\mathcal{Z}, \langle s \rangle)$, by direct calculation it is given by $R^*_\alpha = \langle \mathbf{x} \rangle^{-2\alpha} R^t \langle s \rangle^{2\alpha-1}$. In particular, we obtain many bounded self-adjoint realizations of R on a continuous family of weighted L^2 spaces:*

$$\langle \mathbf{x} \rangle^{-2\alpha} R^t \langle s \rangle^{2\alpha-1} R\colon L^2(\mathbb{R}^2, \langle \mathbf{x} \rangle^{2\alpha}) \to L^2(\mathbb{R}^2, \langle \mathbf{x} \rangle^{2\alpha}), \qquad \alpha > 1/2.$$

One may notice that only the case $\alpha = 1/2$ would make the operator R^tR appear. However, the previous proof breaks down for $\alpha = 1/2$.

4 Lecture 4 - Fourier slice theorem and consequences

4.1 The Fourier Slice Theorem

We first describe our Hilbert scales on $\mathbb{R}^2$ and $\mathcal{Z}$. We define the Fourier transform on $\mathbb{R}^2$ as usual:

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} e^{-i\mathrm{x}\cdot\xi} f(\mathrm{x})\ d\mathrm{x}, \qquad \xi \in \mathbb{R}^2, \qquad f \in L^2(\mathbb{R}^2),$$

and on $\mathcal{Z}$,

$$\tilde{g}(\sigma,\theta) = \int_{\mathbb{R}} e^{-is\sigma} g(s,\theta)\ ds, \qquad g \in L^2(\mathcal{Z}).$$

for $r \in \mathbb{R}$, define

$$\|f\|^2_{H^r}(\mathbb{R}^2) = \int_{\mathbb{R}^2} (1+|\xi|^2)^r |\hat{f}(\xi)|^2\ d\xi, \qquad \|g\|^2_{H^s(\mathcal{Z})} = \int_{\mathcal{Z}} (1+\sigma^2)^r |\tilde{g}(\sigma,\theta)|^2\ d\sigma\ d\theta.$$

Theorem 7 (Fourier Slice Theorem)**.** *For all $f \in \mathscr{S}(\mathbb{R}^2)$,*

$$\widetilde{Rf}(\sigma,\theta) = \hat{f}(\sigma\boldsymbol{\theta}), \qquad (\sigma,\theta) \in \mathcal{Z}.$$

Proof.

$$\widetilde{Rf}(\sigma,\theta) = \int_{\mathbb{R}} e^{-i\sigma s} Rf(s,\theta)\ ds = \int_{\mathbb{R}} e^{-i\sigma s} \int_{\mathbb{R}} f(s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp)\ dt\ ds.$$

The result follows by changing variable $\mathrm{x}(s,t) = s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp$, noticing that $s\sigma = \mathrm{x}\cdot\boldsymbol{\theta}\sigma = \mathrm{x}\cdot(\sigma\boldsymbol{\theta})$. □

Some consequences:

- If $f \in \mathscr{S}$ satisfies $Rf = 0$, then $\widetilde{Rf} = 0$ and hence $\hat{f} = 0$, hence $f = 0$ (Injectivity # 2).
- The FST motivates a first reconstruction formula: take Rf, compute its 1D Fourier transform in s, this is the Fourier transform of f in polar coordinates; recover f from its Fourier transform. This last step may require a tricky interpolation from polar to cartesian, and we will see below other reconstruction formulas which do not have this drawback.

4.2 Stability estimates

In the Hilbert scales defined above, the FST allows to formulate sharp stability estimates.

Theorem 8 (two-sided estimates, see Thm 2.2.2. in [Bal12])**.** *Let $f \in H^r(\mathbb{R}^n)$ for some $r \in \mathbb{R}$. Then we have the following inequalities:*

(i) $\sqrt{2}\|f\|_{H^r(\mathbb{R}^2)} \le \|Rf\|_{H^{r+1/2}(\mathcal{Z})}$.

(ii) For any smooth and compactly supported function χ,

$$\|R(\chi f)\|_{H^{r+1/2}(\mathcal{Z})} \le C_\chi \|\chi f\|_{H^r(\mathbb{R}^2)}.$$

Other Hilbert scales were recently defined in [Sha16], allowing further isometric estimates similar to (i).

From the "two-sidedness" of the conclusion above, the estimates can not be improved on the Sobolev scale. This is an embodiment of the fact that the problem is "ill-posed of order 1/2". Note however that one direction does require the use of a cutoff function χ, and the constant C_χ cannot be made independent of χ.

Proof. For the purpose of proving both (i) and (ii), we first compute, as a preliminary:

$$\|Rf\|^2_{H^{r+\frac{1}{2}}(\mathcal{Z})} = \int_{\mathbb{R}\times\mathbb{S}^1} |\widetilde{Rf}(\sigma,\theta)|^2 \langle\sigma\rangle^{2r+1}\ d\sigma\ d\theta = 2\int_{(0,\infty)\times\mathbb{S}^1} |\widetilde{Rf}(\sigma,\theta)|^2 \langle\sigma\rangle^{2r+1}\ d\sigma\ d\theta,$$

where we have used the symmetry $\widetilde{Rf}(\sigma,\theta) = \widetilde{Rf}(-\sigma,\theta+\pi)$. We now use the Fourier Slice Theorem and change variable $\xi = \sigma\boldsymbol{\theta}$ to arrive at

$$\|Rf\|^2_{H^{r+\frac{1}{2}}(\mathcal{Z})} = 2\int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \langle\xi\rangle^{2r} \frac{\langle\xi\rangle}{|\xi|}\ d\xi.$$

Now notice that we *always* have $\langle\xi\rangle \ge |\xi|$, so estimate (i) following immediately from bounding $\frac{\langle\xi\rangle}{|\xi|}$ from below by 1.

To obtain (ii) however, we see that there is an issue[18] because $\frac{\langle\xi\rangle}{|\xi|}$ is not bounded on $\mathbb{R}^2$. Fixing a "cutoff" function $\chi \in C_c^\infty(\mathbb{R}^2)$, we rewrite the last equation as

$$\frac{1}{2}\|R(\chi f)\|^2_{H^{r+\frac{1}{2}}(\mathcal{Z})} = \int_{\mathbb{R}^2} |\widehat{\chi f}(\xi)|^2 \langle\xi\rangle^{2r} \frac{\langle\xi\rangle}{|\xi|}\ d\xi = \underbrace{\int_{|\xi|<1}}_{I_1} + \underbrace{\int_{|\xi|\ge 1}}_{I_2}.$$

On the term I_2, for $|\xi| \ge 1$, we always have $\frac{\langle\xi\rangle}{|\xi|} = \left(\frac{1}{|\xi|^2}+1\right)^{1/2} \le \sqrt{2}$, and hence $I_2 \le \sqrt{2}\|\chi f\|_{H^r}$. Bounding I_1 is the trickier part. We exploit the fact that the singularity at $\xi = 0$ is integrable:

$$I_1 = \int_{|\xi|\le 1} |\widehat{\chi f}(\xi)|^2 \langle\xi\rangle^{2r} \frac{\langle\xi\rangle}{|\xi|}\ d\xi \le \underbrace{\int_{|\xi|\le 1} \langle\xi\rangle^{2r} \frac{\langle\xi\rangle}{|\xi|}\ d\xi}_{=C<\infty} \cdot \sup_{|\xi|\le 1} |\widehat{\chi f}(\xi)|^2.$$

Now fix $\psi \in C_c^\infty(\mathbb{R}^2)$ equal to 1 on the support of χ so that $\chi f = \psi\chi f$. Then for $|\xi| \le 1$,

$$\begin{aligned}
|\widehat{\chi f}(\xi)|^2 = |\widehat{\chi f \psi}(\xi)|^2 &= \left|\int_{\mathbb{R}^2} \chi(\mathrm{x}) f(\mathrm{x}) \psi(\mathrm{x}) e^{-i\mathrm{x}\cdot\xi}\ d\mathrm{x}\right|^2 \\
&= \frac{1}{(2\pi)^2}\left|\int_{\mathbb{R}^2} \widehat{\chi f}(\eta)\ \widehat{\psi e^{-i(\cdot)\cdot\xi}}(\eta)\ d\eta\right|^2 \qquad \text{(Parseval)} \\
&\le \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} |\widehat{\chi f}(\eta)|^2 \langle\eta\rangle^{2r}\ d\eta \cdot \int_{\mathbb{R}^2} |\widehat{\psi e^{-i(\cdot)\cdot\xi}}(\eta)|^2 \langle\eta\rangle^{-2r}\ d\eta \qquad \text{(Cauchy-Schwarz)} \\
&\le \frac{1}{(2\pi)^2}\|\chi f\|^2_{H^r} \int_{\mathbb{R}^2} |\widehat{\psi}(\eta+\xi)|^2 \langle\eta\rangle^{-2r}\ d\eta.
\end{aligned}$$

[18] This is sometimes referred to as a zero-frequency problem, since it occurs at $\xi = 0$.

The last factor is uniformly bounded over the set $|\xi| \le 1$. Putting everything back together,

$$I_1 \le C_\chi \|\chi f\|_{H^r},$$

where the constant C_χ may depend on the size of the support of f (through the choice of function ψ). □

Some comments about the H^s spaces on $\mathcal{Z}$. The classical Sobolev scale on $\mathcal{Z}$ should read something like: $H^k_{\text{cl}}(\mathcal{Z})$ is made of functions g in $L^2(\mathcal{Z})$ such that for all $\ell, m \in \mathbb{N}_0$ with $\ell + m \le k$, $\partial_\theta^\ell \partial_s^m g$ (in the sense of distributions) belongs to $L^2(\mathcal{Z})$.

Our current definition only allows regularity with respect to s, and as such, defines a family of "anisotropic Sobolev spaces". What, then, makes us recover regularity in the θ variable when a function has s-regularity ? Usually, nothing. However in our case, the constraint that g be in the range of R will allow one to propage regularity in s into regularity in θ.

In a nutshell, the topologies $H^k(\mathcal{Z})$ and $H^k_{\text{cl}}(\mathcal{Z})$ are not equivalent in general, but they are on the range of R (when considered on functions with compact support). Such a statement finds rigorous formulations in, e.g., [Nat01, Ch. II, Th. 5.2] and [Mon20, Proposition 16].

4.3 Interlude: Riesz potentials, Hilbert transform and convolution operators

This section serves as preliminary for the one that follows. Some details have been skipped in lecture.

Recall the identity $\widehat{f \star g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$. It's true for $f, g \in \mathscr{S}(\mathbb{R}^n)$ but can be pushed to $g \in \mathscr{S}'$ and f a distribution with compact support.

Given f a distribution with compact support[19], one may consider the convolution operator $A_f \colon \mathscr{S} \to \mathscr{S}'$ defined by $A_f(g) := f \star g$. The previous identity tells us that it can be computed in two ways: (i) by direct computation of the convolution, or (ii) via Fourier transform, as $A_f = \mathcal{F}^{-1} \circ \hat{f}(\xi) \circ \mathcal{F}$. In other context, the function $\hat{f}(\xi)$ is called the *symbol* (or Fourier multiplier) of A_f, and A_f is called the *quantization* of $\hat{f}$. The form (ii) is extremely useful for computing purposes, as the computation of $\mathcal{F}$ and $\mathcal{F}^{-1}$ can be done using the *Fast Fourier Fransform* (FFT).

Several operators are convolution operators in disguise:

- The identity is nothing but $g \mapsto \delta \star g$, which, as a Fourier multiplier, gives $\hat{f}(\xi) = 1$. In particular, $\hat{\delta} = 1$.
- Any differential operator $P(D) = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha$ can be viewed as convolution by $P(D)\delta$, with Fourier multilplier $P(i\xi)$. In particular, the Laplacian Δ has symbol $-|\xi|^2$.

A class of interest to us in the next section will be the **Riesz potentials**: on $\mathbb{R}^n$, for $\alpha < n$, define I^α via Fourier transform:

$$\widehat{I^\alpha f}(\xi) := |\xi|^{-\alpha} \hat{f}(\xi). \tag{31}$$

[19] An element of $\mathscr{E}'$ in the PDE notes.

The condition $\alpha < n$ ensures that $|\xi|^{-\alpha}\hat{f}(\xi)$ remains locally integrable so that it can be viewed as an element of $\mathscr{S}'$.

Some comments: for $\alpha < 0$, I^α is "unsmoothing", for example, $I^{-2} = -\Delta$; for $\alpha = 0$, I^0 is the identity; for $0 < \alpha < n$, I^α is smoothing. It is also immediately clear that $I^\alpha \circ I^\beta = I^\beta \circ I^\alpha = I^{\alpha+\beta}$ as long as α, β and $\alpha+\beta$ are strictly less than n.

The Hilbert transform. Picking up where we left off, let us focus on the one-dimensional case, using s for the physical variable and σ for its dual Fourier variable. The operator I^{-1} is then a convolution operator with symbol $|\sigma|$, and one may wonder how close it is from being a $\frac{d}{ds}$ derivative, an operator whose symbol is $i\sigma$. The answer is fairly simple: upon writing

$$|\sigma| = i\sigma \cdot \frac{1}{i}\mathrm{sgn}(\sigma),$$

we see that I^{-1} can be written as the product of two commuting operators, one being $\frac{d}{ds}$, and the other one being the operator with Fourier multiplier $\hat{h}(\sigma) := \frac{1}{i}\mathrm{sgn}(\sigma)$. We call that operator the **Hilbert transform**. Namely, define

$$H\colon L^2(\mathbb{R}) \to L^2(\mathbb{R}), \qquad Hf := \mathcal{F}^{-1}(\hat{h}(\xi)\mathcal{F}f). \tag{32}$$

From this definition, some interesting properties follow quickly: Parseval's formula implies that H is an isometry of L^2; moreover, $H^2 = -Id$.

One may however wonder what that operator looks like as a convolution operator. Namely: if $\hat{h}(\sigma) := \frac{1}{i}\mathrm{sgn}(\sigma)$, what does $h \in \mathscr{S}'$ look like ? Upon defining the distribution $p.v.\frac{1}{s}$ by

$$\langle p.v.\frac{1}{s}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} := \lim_{\varepsilon\to 0}\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{\varphi(s)}{s}\ ds = \int_0^\infty \frac{\varphi(s)-\varphi(-s)}{s}\ ds,$$

we have the following

Lemma 9. *With $\hat{h}(\sigma) = \frac{1}{i}sgn(\sigma)$, we have $h(s) = \frac{1}{\pi}p.v.\frac{1}{s}$.*

Proof. We first show that $\frac{d}{d\sigma}\mathrm{sgn}(\sigma) = 2\delta$. Indeed, for any $\varphi \in \mathscr{S}$,

$$\begin{aligned}\langle\frac{d}{d\sigma}\mathrm{sgn}(\sigma), \varphi\rangle = -\langle \mathrm{sgn}(\sigma), \varphi'\rangle &= -\int_{\mathbb{R}} \mathrm{sgn}(\sigma)\varphi'(\sigma)\ d\sigma \\ &= \int_{-\infty}^0 \varphi'(\sigma)\ d\sigma + \int_0^\infty \varphi'(\sigma)\ d\sigma = 2\varphi(0) = \langle 2\delta, \varphi\rangle.\end{aligned}$$

In addition, we claim that $\delta = \widehat{\frac{1}{2\pi}}$, as the line below shows:

$$\langle \delta, \varphi\rangle = \varphi(0) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\varphi}(\xi)\ d\xi = \langle \frac{1}{2\pi}, \hat{\varphi}\rangle = \langle \widehat{\frac{1}{2\pi}}, \varphi\rangle.$$

Combining the past two claims, we arrive at the conclusion that

$$\widehat{-ish}(\sigma) = \frac{d}{d\sigma}\hat{h}(\sigma) = \frac{d}{d\sigma}\frac{1}{i}\mathrm{sgn}(\sigma) = \frac{2}{i}\delta = \frac{2}{i}\widehat{\frac{1}{2\pi}}.$$

Since the Fourier transform is an isomorphism, we deduce the equality of tempered distributions:

$$sh(s) = \frac{1}{\pi}. \tag{33}$$

One may be tempted to divide by s and call it a day, but the function $\frac{1}{s}$ does not define a distribution. However, the distribution $\frac{1}{\pi}p.v.\frac{1}{s}$ does, and in fact, solves equation (33) (exercise: check this). As a result, we have the following equality of tempered distributions

$$s\left(h(s) - \frac{1}{\pi}p.v.\frac{1}{s}\right) = 0.$$

We now use the following lemma, whose sketch is given in Ex. 19:

Lemma 10. *If $u \in \mathscr{S}'$ solves $su = 0$ in $\mathscr{S}'$, then $u = C\delta$ for some constant C.*

We then deduce that

$$h(s) = \frac{1}{\pi}p.v.\frac{1}{s} + C\delta.$$

Finally, C is zero by evenness/oddness considerations: h and $\frac{1}{\pi}p.v.\frac{1}{s}$ are both odd in the sense that $\langle h, \varphi(-s)\rangle = -\langle h, \varphi\rangle$ for all $\varphi \in \mathscr{S}$, and δ is even in the sense that $\langle \delta, \varphi(-s)\rangle = \langle \delta, \varphi\rangle$ for all $\varphi \in \mathscr{S}$. The last display in the equality forces C to be zero. Hence the result. $\square$

By virtue of Lemma (9), we obtain the following second characterization of the Hilbert transform:

$$Hf(t) = \frac{1}{\pi}p.v.\int_{\mathbb{R}} \frac{f(s)}{t-s}\ ds, \tag{34}$$

which we will use later.

Exercise 19. *Sketch of proof of Lemma 10: suppose $su = 0$. Fix $\chi \in C_c^\infty(\mathbb{R})$ a function equal to 1 in a neighbourhood of 0. For $\varphi \in \mathscr{S}$, write $\varphi(s) = \varphi(0)\chi(s) + s\psi(s)$ for some $\psi \in \mathscr{S}$. Conclude by computing $\langle u, \varphi\rangle$ using this decomposition.*

Exercise 20 (A second proof of Lemma 9).

1. *Show that $\frac{s}{\epsilon^2+s^2}$ converges to $p.v.\frac{1}{s}$ in $\mathscr{S}'$.*
2. *Compute the inverse Fourier transform of the L^1 function $\widehat{h_\epsilon}(\sigma) = \frac{1}{i}sgn(\sigma)e^{-\epsilon|\sigma|}$. Conclude by $\mathscr{S}'$-approximation and continuity of the Fourier transform in $\mathscr{S}'$.*

4.4 Filtered-Backprojection formulas

Exact formulas. In what follows, we will somewhat abuse notation: for a function on $\mathbb{R}^2$, we will define the Riesz potential $I^\alpha f$ as in (31). For functions on $\mathcal{Z}$, we will use the 1D Riesz potential w.r.t. the s variable, namely

$$\widetilde{I^\alpha g}(\sigma,\theta) := |\sigma|^{-\alpha}\tilde{g}(\sigma,\theta).$$

Using these operators, we can then derive the following one-parameter family of inversion formulas, see also [Nat01, Thm. 2.1]. Recall the definition of the backprojection operator

$$R^t g(\mathrm{x}) = \int_{\mathbb{S}^1} g(\mathrm{x}\cdot\boldsymbol{\theta},\theta)\ d\theta, \qquad \mathrm{x} \in \mathbb{R}^2.$$

Theorem 11. *For all $\alpha < 2$ and $f \in \mathscr{S}(\mathbb{R}^2)$,*

$$f(x) = \frac{1}{4\pi} I^{-\alpha} R^t I^{\alpha-1} Rf.$$

Proof. We compute, using the Fourier inversion formula

$$\begin{aligned}
I^\alpha f(\mathrm{x}) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{I^\alpha f}(\xi) e^{i\mathrm{x}\cdot\xi}\ d\xi \\
&\overset{(a)}{=} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\xi|^{-\alpha} \hat{f}(\xi) e^{i\mathrm{x}\cdot\xi}\ d\xi \\
&= \frac{1}{(2\pi)^2} \int_0^\infty \int_{\mathbb{S}^1} |\sigma|^{1-\alpha} \hat{f}(\sigma\boldsymbol{\theta}) e^{i\mathrm{x}\cdot\sigma\boldsymbol{\theta}}\ d\sigma\ d\theta \qquad (\xi = \sigma\boldsymbol{\theta}) \\
&\overset{(b)}{=} \frac{1}{(2\pi)^2} \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{S}^1} |\sigma|^{1-\alpha} \widetilde{Rf}(\sigma,\theta) e^{i\mathrm{x}\cdot\sigma\boldsymbol{\theta}}\ d\sigma\ d\theta \\
&= \frac{1}{4\pi} \int_{\mathbb{S}^1} I^{1-\alpha} Rf(\mathrm{x}\cdot\boldsymbol{\theta}, \theta)\ d\theta \\
&= \frac{1}{4\pi} R^t I^{1-\alpha} Rf(\mathrm{x}).
\end{aligned}$$

In (a) we have used the definition of I^α, and in (b) we have used the symmetry $\widetilde{Rf}(-\sigma, \theta+\pi) = \widetilde{Rf}(\sigma,\theta)$ to extend the integral to $\mathbb{R}$. □

Special cases:

- The case $\alpha = 1$ reads

$$f = \frac{1}{4\pi} I^{-1} R^t Rf,$$

  where I^{-1} corresponds to the Fourier multiplier $|\xi|$, i.e. $I^{-1} = \sqrt{-\Delta}$. We could have guessed this from the work that was done in Section 3.

- Perhaps the most popular case is when $\alpha = 0$, this gives the celebrated *filtered-backprojection formula*:

$$f = \frac{1}{4\pi} R^t I^{-1} Rf. \tag{35}$$

  The operator I^{-1} is the one-dimensional Fourier multiplier $|\sigma|$. Computing it is done columnwise (for each θ separately), and each column is processed via `fast fourier transform` before and after multiplication of by $|\sigma|$. As explained above we can write $I^{-1} = \frac{d}{ds} H = H \frac{d}{ds}$.

- One can rewrite (35) so as to recover Radon's original inversion formula:

$$f(\mathrm{x}) = \frac{-1}{\pi} \int_0^\infty \frac{dF_{\mathrm{x}}(q)}{q}, \qquad F_{\mathrm{x}}(q) := \frac{1}{2\pi} \int_{\mathbb{S}^1} Rf(q + \mathrm{x}\cdot\boldsymbol{\theta}, \theta)\ d\theta.$$

  This formula has a nice geometric interpretation: $f(\mathrm{x})$ is a weighted functional of $F_{\mathrm{x}}(q)$, which is the average of Rf over all lines tangent to the circle of center x and radius q.

Approximate formulas. The Fourier multiplier $|\sigma|$, also called a *ramp filter*, amplifies high frequencies more than low frequencies. Since actual images have limited bandwidth[20], and since noise tends to be more prominent at high frequencies, we want to replace this "exact" filter by a low-pass one, emphasizing the features of f which can be more faithfully reconstructed by the data at hand.

The general setting is the following: take $w \in \mathscr{S}(\mathbb{R})$, $w \equiv w(s)$ an even function (in the sense that $w(-s) = w(s)$), and let $W = R^t w$, that is to say

$$W(\mathrm{x}) = \int_{\mathbb{S}^1} w(\mathrm{x}\cdot\boldsymbol{\theta})\ d\theta = W(|\mathrm{x}|).$$

Then following the same scheme of proof as Theorem 11, one can show the following general result. In the statement $\overset{\mathrm{x}}{\star}$ denotes two-dimensional convolution and $\overset{s}{\star}$ denotes one-dimensional s-convolution.

Theorem 12 (Filtered-Backprojection Formulas)**.**

$$W \overset{\mathrm{x}}{\star} f = \frac{1}{4\pi} R^t (w \overset{s}{\star} Rf). \tag{36}$$

Exercise 21. *Prove Theorem 12.*

Remark 3 (On filters)**.** *Formulas of this type do not reconstruct f exactly, rather, they reconstruct $W \star f$, which should be thought of as a "regularized", low-frequency version of the unknown.*

The relevant parameter is $\tilde{w}(\sigma)$, which should be in some sense a low-pass version of $|\sigma|$. Fixing a cutoff bandwidth $b > 0$, here are the most commonly used filters in Matlab's `iradon` *function.*

`Ram-Lak`: $\tilde{w}_b(\sigma) = |\sigma|\chi_{[-b,b]}(\sigma)$.

`Shepp-Logan`: $\tilde{w}_b(\sigma) = |\sigma|\chi_{[-b,b]}(\sigma)\frac{\sin(\pi\sigma/2b)}{\pi\sigma/2b}$.

`cosine`: $\tilde{w}_b(\sigma) = |\sigma|\chi_{[-b,b]}(\sigma)\cos(\frac{\pi}{2}\frac{|\sigma|}{b})$.

`Hahn`: $\tilde{w}_b(\sigma) = |\sigma|\chi_{[-b,b]}(\sigma)(.54 + .46\cos(\pi\sigma/b))$.

`Hamming`: $\tilde{w}_b(\sigma) = |\sigma|\chi_{[-b,b]}(\sigma)(.5 + .5\cos(\pi\sigma/b))$.

Exercise 22. *The Radon transform in higher dimensions. For $f \in \mathscr{S}(\mathbb{R}^n)$, define*

$$Rf(s,\omega) = \int_{\{x\cdot\omega = s\}} f, \qquad s\in\mathbb{R},\ \omega\in\mathbb{S}^{n-1},$$

where the hyperplane $h_{s,\omega} := \{x\in\mathbb{R}^n,\ x\cdot\omega = s\}$ is equipped with its natural Lebesgue measure. Notice the symmetry $Rf(s,\omega) = Rf(-s,-\omega)$. Define $\hat{f}$ the Fourier transform as usual on $\mathbb{R}^n$, and $\tilde{g}(\sigma,\omega)$ the one-dimensional Fourier transform along the s factor for functions on $\mathbb{R}\times\mathbb{S}^{n-1}$.

1. *Prove the Fourier slice theorem: for any $f \in \mathscr{S}(\mathbb{R}^n)$, $\widetilde{Rf}(\sigma,\omega) = \hat{f}(\sigma\omega)$,*

[20] i.e., their Fourier transforms is supported in a ball of radius b. The smallest such b is often called the bandwidth.

2. *Compute the formal adjoint operator of $R : L^2(\mathbb{R}^n) \to L^2(\mathbb{R} \times \mathbb{S}^{n-1})$, call it R^*. What is its geometric meaning ?*

3. *Recalling the definition of the Riesz potentials as usual (for $\alpha < n$, $\widehat{I^\alpha f}(\xi) = |\xi|^{-\alpha}\hat{f}(\xi)$), prove the filtered-backprojection formula, true for any $f \in \mathscr{S}(\mathbb{R}^n)$:*

$$f = \frac{1}{2}(2\pi)^{1-n} I^{-\alpha} R^* I^{\alpha-n+1} Rf, \qquad \alpha < n.$$

4. *Focus on the case $\alpha = 0$ and explain how the locality of I^{-n+1} depends on the parity of n.*

5. *Conclude that in odd dimensions, $f(x)$ can be reconstructed from the Radon transform over all planes intersecting a small neighborhood of x.*

Exercise 23. *Radon transform and wave equation. Recall d'Alembert's formula*

$$v(x,t) = \frac{1}{2}(f(x-t) + f(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} g(u)\ du$$

which provides the expression for the unique solution to the $1+1$-dimensional wave problem

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} = 0 \qquad (x \in \mathbb{R},\ t > 0), \qquad v|_{t=0} = f, \qquad \frac{\partial v}{\partial t}|_{t=0} = g.$$

This problem shows how the Radon transform and d'Alembert's formula provide a method for solving wave equations in $\mathbb{R}^n \times (0,\infty)$ for any $n \in \mathbb{N}$.

1. *Prove that for $f \in \mathscr{S}(\mathbb{R}^n)$, $\frac{d^2}{ds^2} Rf = R[\Delta f]$.*

2. *Use the previous result to derive a solution of the wave problem*

$$\partial_{tt} u - \Delta u = 0 \qquad (\mathbf{x} \in \mathbb{R}^n, t > 0), \qquad u(\mathbf{x}, 0) = u_0(\mathbf{x}), \qquad \partial_t u(\mathbf{x}, 0) = u_1(\mathbf{x}),$$

by setting up a PDE problem for the function $v(s,\omega,t) := R[u(\cdot,t)](s,\omega)$ (where the Radon transform acts on the $\mathbf{x}$-variable only).

4.5 Moving to functions supported on the unit disk

As a precursor to the next lecture, we shall now restrict functions to those with compact support in the unit disk $\mathbb{D}$, and make the following observation: the Radon transform is not only $L^2(\mathbb{D}) \to L^2([-1,1] \times \mathbb{S}^1, ds\ d\theta)$-bounded, but the presence of short curves allows us to shrink the co-domain immediately, to write:

Theorem 13. *The Radon transform is bounded as an operator*

$$R\colon L^2(\mathbb{D}) \to L^2([-1,1] \times \mathbb{S}^1, (1-s^2)^{-1/2}\ ds\ d\theta). \tag{37}$$

Exercise 24. *Prove Theorem 13.*

In the next lecture, we will switch from parallel coordinates (s, θ) to fan-beam coordinates

$$(s, \theta) \mapsto (\beta = \theta - \pi/2 - \sin^{-1} s, \alpha = \sin^{-1} s) \in \mathbb{S}^1 \times (-\pi/2, \pi/2). \tag{38}$$

The latter coordinates, arising from the idea of parameterizing all segments passing through $\mathbb{D}$ from the boundary, are more amenable to a theory of X-ray transforms on Riemannian manifolds, since parallel geometry is somewhat a "global" view of geodesics which one may not have on general manifolds. In addition, this change of variable conveniently desingularizes the volume form $(1 - s^2)^{-1/2}\, ds\, d\theta$ into $d\alpha\, d\beta$.

5 Lecture 5 - The X-ray transform on the unit disk $\mathbb{D}$

We will now return to an example where the integral geometric operator of interest has discrete spectrum, and where a Singular Value Decomposition can be derived.

5.1 From parallel beam to fan-beam

From Exercise (24), one may find out that the operator

$$R\colon L^2(\mathbb{D}) \to L^2([-1,1]\times\mathbb{S}^1, (1-s^2)^{-1/2}\ ds\ d\theta)$$

is bounded. In spirit this weight on the codomain arises from the fact that integration curves become shorter as $|s|$ approaches 1 and this also contributes to the smallness of Rf there. Note that we have shrunk the co-domain, and thus we have changed the expression of the adjoint, which now looks like

$$R^* g(\mathrm{x}) = \int_{\mathbb{S}^1} [(1-s^2)^{-1/2} g](\mathrm{x}\cdot\boldsymbol{\theta}, \theta)\ d\theta = \int_{\mathbb{S}^1} \frac{g(\mathrm{x}\cdot\boldsymbol{\theta},\theta)}{(1-(\mathrm{x}\cdot\boldsymbol{\theta})^2)^{1/2}}\ d\theta.$$

There are "natural" angle variables in which the measure $(1-s^2)^{-1/2}\ ds\ d\theta$ becomes $d\alpha\ d\beta$, called **fan-beam coordinates**: $\beta\in\mathbb{S}^1$ parameterizes a point from the boundary, while $\alpha\in(-\pi/2,\pi/2)$ parameterizes the angle where to cast the line segment, relative to the inner pointing normal. See Fig. 1 for an example of both transforms side-by-side.

We now call I_0 the operator $R|_{L^2(\mathbb{D})}$ to connect it with X-ray transform notation[21]. Hence, for $f\in L^2(\mathbb{D})$, let us define

$$I_0 f(\beta,\alpha) := \int_0^{2\cos\alpha} f(e^{i\beta} + te^{i(\beta+\pi+\alpha)})\ dt, \qquad (\beta,\alpha)\in\mathbb{S}^1\times[-\pi/2,\pi/2]. \tag{39}$$

Below, we will denote $\partial_+ S\mathbb{D} = \mathbb{S}^1_\beta \times [-\pi/2,\pi/2]_\alpha$. This notation, customarily used in the context of X-ray transforms on manifolds, means the "inward-point boundary of the unit tangent bundle $S\mathbb{D}$", i.e. the set of boundary points of $\mathbb{D}$ together with the inward-pointing unit tangent vectors. One may also think of the outward-pointing vectors $\partial_- S\mathbb{D} = \mathbb{S}^1_\beta\times[\pi/2,3\pi/2]_\alpha$, and we have $\partial S\mathbb{D} = \partial_+ S\mathbb{D}\cup\partial_- S\mathbb{D}$.

Elementary mapping properties. A first observation is that the symmetry (27) translates in this case into

$$I_0 f(\beta,\alpha) = I_0 f(\beta+\pi+2\alpha, -\alpha), \qquad (\beta,\alpha)\in\partial_+ S\mathbb{D}. \tag{40}$$

There is another interesting observation: upon changing variable $t = (2\cos\alpha)u$ inside the integral, one may rewrite this integral as:

$$I_0 f(\beta,\alpha) := 2\cos\alpha \int_0^1 f\left(e^{i(\alpha+\beta)}((1-2u)\cos\alpha - i\sin\alpha)\right)\ du, \qquad (\beta,\alpha)\in\mathbb{S}^1\times[-\pi/2,\pi/2]. \tag{41}$$

[21] On (M,g) a Riemannian manifolds with unit tangent bundle SM, I is often notation for the geodesic X-ray transform on functions on SM, while I_0 is its restriction to functions on M.

Exercise 25. *Show* (41).

If $f \in C^\infty(\mathbb{D})$, we see that $I_0 f$ is not only smooth on the interior of $\partial_+ S\mathbb{D}$, but it naturally wants to be extended to $\partial S\mathbb{D} := \mathbb{S}^1_\beta \times \mathbb{S}^1_\alpha$ into a smooth function that is odd on the "circle fibers of $\partial S\mathbb{D}$", i.e. odd with respect to the involution $(\beta,\alpha) \mapsto (\beta, \alpha+\pi)$. Combining this last comment with the fact that the composition of $(\beta,\alpha) \mapsto (\beta+\pi+2\alpha, -\alpha)$ with $(\beta,\alpha)\mapsto(\beta,\alpha+\pi)$ is the "scattering relation"

$$\partial_+ S\mathbb{D} \ni (\beta,\alpha) \mapsto (\beta+\pi+2\alpha, \pi-\alpha) \in \partial_- S\mathbb{D}, \tag{42}$$

we arrive at the conclusion that

Lemma 14. *The operator I_0 maps $C^\infty(\mathbb{D})$ into $C^\infty_{\alpha,-,+}(\partial_+ S\mathbb{D})$, the space of smooth functions on $\partial_+ S\mathbb{D}$ satisfying* (40)*, and whose odd extension to $\partial_- S\mathbb{D}$ by oddness via the map* (42) *is smooth on $S\mathbb{D}$.*

See [MM21] for more general facts and motivation behind the space $C^\infty_{\alpha,-,+}(\partial_+ S\mathbb{D})$.

The adjoint I_0^*. Since $I_0(\beta,\alpha) = Rf(\sin\alpha, \beta+\pi/2+\alpha)$, using a change of variable from (s,θ) to (β,α), one may show that $I_0\colon L^2(\mathbb{D}) \to L^2(\mathbb{S}^1_\beta \times [-\pi/2,\pi/2]_\alpha,\ d\alpha\ d\beta)$ is bounded, and its adjoint (a.k.a. *the backprojection operator*) takes the form

$$I_0^* g(\mathrm{x}) = \int_{\mathbb{S}^1} \frac{1}{\cos\alpha_-(\mathrm{x},\theta)} g(\beta_-(\mathrm{x},\theta), \alpha_-(\mathrm{x},\theta))\ d\theta, \tag{43}$$

where $\beta_-(\mathrm{x},\theta)$, $\alpha_-(\mathrm{x},\theta)$ are the fan-beam coordinates of the unique line passing through (x,θ), as explained in section 5.3 below.

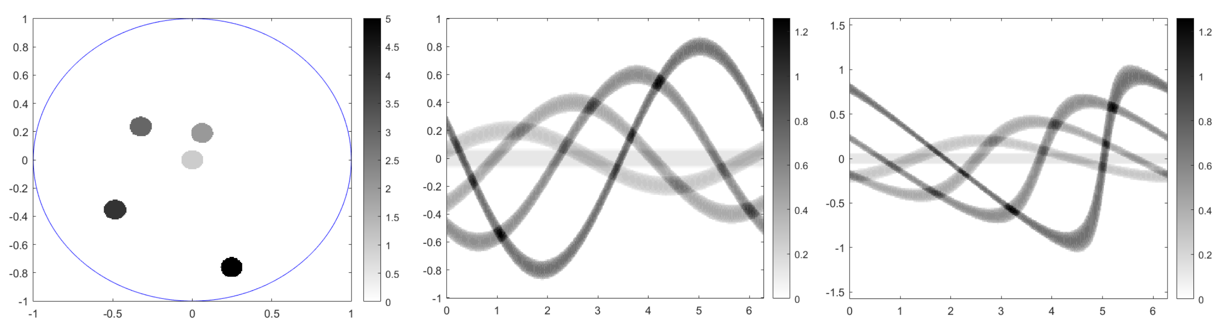

Figure 1: left to right: a function f, its Radon transform Rf (axes: (θ,s)), its X-ray transform $I_0 f$ (axes: (β,α))

The purpose below is to compute the full SVD of I_0 using the method of intertwining differential operators, which we now recall.

5.2 The method of intertwining operators

Suppose we have a bounded, injective operator between two Hilbert spaces

$$A\colon (\mathcal{H}_1, \|\cdot\|_1) \to (\mathcal{H}_2, \|\cdot\|_2),$$

and suppose we have two operators $D_1\colon \mathcal{D}(D_1) \to \mathcal{H}_1$ and $D_2\colon \mathcal{D}(D_2) \to \mathcal{H}_2$, defined on dense subspaces $\mathcal{D}(D_1) \subset \mathcal{H}_1$ and $\mathcal{D}(D_2) \subset \mathcal{H}_2$, and self-adjoint in the sense that for all $u, v \in \mathcal{D}(D_1)$, $\langle D_1 u, v\rangle_1 = \langle u, D_1 v\rangle_1$, similarly for D_2.

Suppose that the *intertwining relation* $A \circ D_1 = D_2 \circ A$ holds on $\mathcal{D}(D_1)$, and assume further that D_1 has simple spectrum $\lambda_0 \le \lambda_1 \le \dots \le \lambda_n \le \dots$ with eigenvectors $\{u_n\}_{n\ge 0}$, a complete orthonormal set in $\mathcal{H}_1$.

Theorem 15. *With the assumptions above, the SVD of $A\colon \mathcal{H}_1 \to A(\mathcal{H}_1)$ is given by*

$$\left(\frac{u_n}{\|u_n\|_1}, \frac{Au_n}{\|Au_n\|_2}, \frac{\|u_n\|_1}{\|Au_n\|_2}\right)_{n\ge 0}.$$

Note that A is not always surjective, this is why the operator is co-restricted to its range in the statement. If A is not injective, it may be that $Au_n = 0$ for some n, in which case those u_n's participate in the kernel of A.

Proof of Theorem 15. For all $n \ge 0$, set $v_n = Au_n$. We have that

$$D_2 v_n = D_2(Au_n) = A(D_1 u_n) = A(\lambda_n u_n) = \lambda_n v_n,$$

hence the family $\{v_n\}_{n\ge 0}$, being eigenvectors with distinct eigenvalues of the self-adjoint operator, is an orthogonal family. By definition, $\{v_n/\|v_n\|_2\}_{n\ge 0}$ is a complete orthonormal set in $A(\mathcal{H}_1)$, and we obviously have

$$A\frac{u_n}{\|u_n\|_1} = \frac{\|v_n\|_2}{\|u_n\|_1}\frac{v_n}{\|v_n\|_2}, \qquad n \ge 0.$$

It just remains to show that

$$A^*\frac{v_n}{\|v_n\|_2} = \frac{\|v_n\|_2}{\|u_n\|_1}\frac{u_n}{\|u_n\|_1}, \qquad n \ge 0.$$

To do this, we simply expand $A^* v_n = \sum_{p\ge 0} a_{np} u_p$, where

$$a_{np} = \frac{\langle A^* v_n, u_p\rangle}{\|u_p\|_1^2} = \frac{\langle v_n, Au_p\rangle}{\|u_p\|_1^2} = \frac{\langle v_n, v_p\rangle}{\|u_p\|_1^2} = \delta_{np}\frac{\|v_n\|_2^2}{\|u_n\|_1^2},$$

and the result follows. □

The original idea can be found in [Maa91], where the backprojection operator I_0^* (in fact, in general dimension there, the adjoint of the Radon transform) is first diagonalized using spherical harmonics, into countably many one-dimensional operators (one for each spherical mode). Each such operator intertwines two second-order differential operators, and the computation of the spectrum of one involves solving ODEs, which in spirit is much simpler than solving integral equations.

What if A is not injective ? Note further that upon taking adjoints and using the self-adjointness of D_1, D_2, we may obtain a second intertwining relation $D_1 \circ A^* = A^* \circ D_2$, and combining the two, we arrive that the fact that $[A^*A, D_1] = 0$ and $[AA^*, D_2] = 0$.

In particular, the eigenspaces of D_1 are A^*A-stable, and since they are all one-dimensional $A^*Au_n = a_n u_n$ for every $n \ge 0$. The kernel of A is precisely the span of those vectors u_n for which $a_n = 0$, and upon removing those and replacing A by its injective restriction $A|_{\{\ker A\}^\perp}$, we can apply Theorem 15.

In what follows, we will **factor in** the circular symmetry of the problem by considering **pairs** of intertwined differential operators.

5.3 Interwiners for the backprojection operator I_0^*

The presentation that follows is a combination of [Mon20, Section 3] and [MM21]. We will denote

$$\partial_+ S\mathbb{D} = \mathbb{S}^1_\beta \times [-\pi/2, \pi/2]_\alpha, \qquad \mu = \cos\alpha.$$

Let us define the operator $I_0^\sharp\colon C_\alpha^\infty(\partial_+ S\mathbb{D}) \to C^\infty(\mathbb{D})$ as the formal adjoint of $I_0 : L^2(\mathbb{D}) \to L^2(\partial_+ S\mathbb{D}, \mu\ d\alpha\ d\beta)$, (so that I_0^* defined in (43) takes the form $I_0^* := I_0^\sharp(\frac{1}{\mu}\cdot)$). Such an operator takes the form

$$I_0^\sharp g(\mathrm{x}) = \int_{\mathbb{S}^1} g(\beta_-(\mathrm{x},\theta), \alpha_-(\mathrm{x},\theta))\ d\theta, \tag{44}$$

where $\beta_-(\mathrm{x},\theta)$, $\alpha_-(\mathrm{x},\theta)$ are the fan-beam coordinates of the unique oriented line passing through (x,θ). In what follows, we will identify x with $\rho e^{i\omega}$. See Figure 2 for a summary.

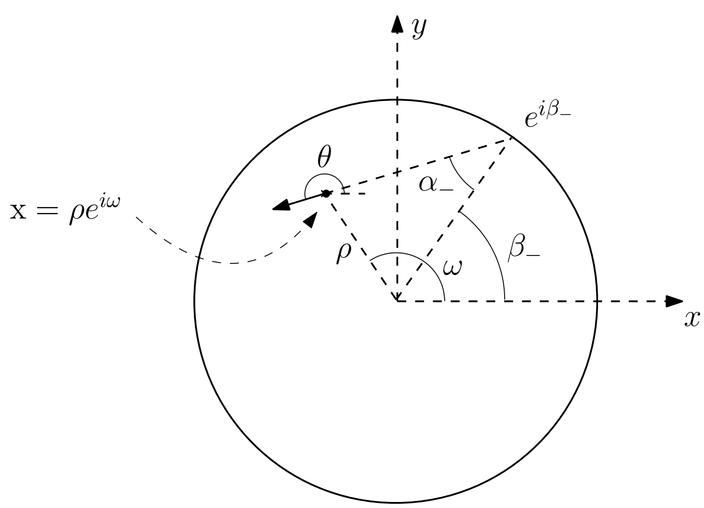

Figure 2: Setting of definition of $(\beta_-(\rho e^{i\omega},\theta), \alpha_-(\rho e^{i\omega},\theta))$ (written as (β_-, α_-) on the diagram). The rotation invariance implies that if θ and ω are translated by δ, then β_- is translated by δ and α_- remains unchanged.

From the observation made in Fig. 2, these functions satisfy the following relation:

$$\beta_-(\rho e^{i\omega}, \theta) = \omega + \beta_-(\rho, \theta - \omega), \qquad \alpha_-(\rho e^{i\omega}, \theta) = \alpha_-(\rho, \theta - \omega).$$

In particular, the expression of $I_0^\sharp g$ immediately becomes

$$I_0^\sharp g(\rho e^{i\omega}) = \int_{\mathbb{S}^1} g(\omega + \beta_-(\rho, \theta - \omega), \alpha_-(\rho, \theta - \omega))\ d\theta = \int_{\mathbb{S}^1} g(\omega + \beta_-(\rho, \theta), \alpha_-(\rho, \theta))\ d\theta.$$

We then immediately see the first intertwining property

$$\partial_\omega \circ I_0^\sharp = I_0^\sharp \circ \partial_\beta, \qquad \partial_\omega \circ I_0^* = I_0^* \circ \partial_\beta.$$

Upon defining

$$T := \partial_\beta - \partial_\alpha, \tag{45}$$

a second intertwining property is then given as follows.

Theorem 16. *Define the operators*

$$L := (1-\rho^2)\frac{\partial^2}{\partial \rho^2} + \left(\frac{1}{\rho} - 3\rho\right)\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \omega^2}, \tag{46}$$

and $D := T^2 + 2\tan\alpha T$. *Then we have the following intertwining properties:*

$$L \circ I_0^\sharp = I_0^\sharp \circ D, \tag{47}$$

$$\mathcal{L} \circ I_0^* = I_0^* \circ (-T^2), \qquad \mathcal{L} := -L + 1. \tag{48}$$

Proof. Proof of (47). In what follows, α_- and β_- will be short for $\alpha_-(\rho,\theta)$ and $\beta_-(\rho,\theta)$. Note the easy two properties

$$\beta_- + \alpha_- + \pi = \theta, \qquad \sin\alpha_- = -\rho\sin\theta.$$

In particular, this gives $\frac{\partial \alpha_-}{\partial \rho} = -\frac{\sin\theta}{\cos\alpha_-} = \frac{1}{\rho}\tan\alpha_-$, $\frac{\partial\alpha_-}{\partial\theta} = \frac{-\rho\cos\theta}{\cos\alpha_-}$, and the derivatives of β_- can be deduced through the relations

$$\frac{\partial \beta_-}{\partial \rho} = -\frac{\partial \alpha_-}{\partial \rho}, \qquad \frac{\partial \beta_-}{\partial \theta} = 1 - \frac{\partial \alpha_-}{\partial \theta}.$$

From these relations, we immediately deduce the property that

$$\frac{\partial}{\partial \rho} I_0^\sharp g = -\frac{1}{\rho} I_0^\sharp[\tan\alpha T g].$$

Iterating this formula, we obtain

$$\frac{\partial^2}{\partial \rho^2} I_0^\sharp g = \frac{1}{\rho^2} I_0^\sharp[\tan\alpha T g] + \frac{1}{\rho^2} I_0^\sharp[\tan\alpha T(\tan\alpha T g)] = \frac{1}{\rho^2} I_0^\sharp[\tan^2\alpha T^2 g - \tan^3\alpha T g].$$

Then by direct algebra, using the last two identities, we obtain

$$\begin{aligned}[(1-\rho^2)\partial_\rho^2 + (\frac{1}{\rho} - 3\rho)\partial_\rho] I_0^\sharp g = \frac{1}{\rho^2} I_0^\sharp[\tan^2\alpha T^2 g - \tan\alpha(1+\tan^2\alpha)Tg]\dots \\ - I_0^\sharp[\tan^2\alpha T^2 g - \tan\alpha(\tan^2\alpha + 3)Tg].\end{aligned} \tag{49}$$

To obtain further identities, we write

$$\begin{aligned} 0 &= \int_{\mathbb{S}^1} \partial_\theta (g(\omega+\beta_-,\alpha_-))\ d\theta \\ &= \int_{\mathbb{S}^1} \left(\frac{\partial \beta_-}{\partial\theta}\partial_\beta + \frac{\partial\alpha_-}{\partial\theta}\partial_\alpha \right) g(\omega+\beta_-,\alpha_-)\ d\theta \\ &= I_0^\sharp[\partial_\beta g] + \rho \int_{\mathbb{S}^1} \frac{\cos\theta}{\cos\alpha_-} Tg(\omega+\beta_-,\alpha_-)\ d\theta, \end{aligned}$$

as well as

$$\begin{aligned} 0 &= \int_{\mathbb{S}^1} \partial_\theta^2 (g(\omega+\beta_-,\alpha_-))\ d\theta \\ &= \int_{\mathbb{S}^1} \partial_\theta \left(\partial_\beta g + \frac{\rho\cos\theta}{\cos\alpha_-} Tg \right)\ d\theta \\ &= \int_{\mathbb{S}^1} \left(\partial_\beta^2 g + \frac{2\rho\cos\theta}{\cos\alpha_-} T\partial_\beta g - \left(\frac{\rho\sin\theta}{\cos\alpha_-} + \rho^2\cos^2\theta \frac{\sin\alpha_-}{\cos^3\alpha_-} \right) Tg + \frac{\rho^2\cos^2\theta}{\cos^2\alpha_-} T^2 g \right)\ d\theta. \end{aligned}$$

From the previous identity and the fact that $T\partial_\beta = \partial_\beta T$, the second term equals $-2I_0^\sharp[\partial_\beta^2 g]$. In the remaining terms, we use that $-\rho\sin\theta = \sin\alpha_-$ and $\rho^2\cos^2\theta = \rho^2(1-\sin^2\theta) = \rho^2 - \sin^2\alpha_-$ and the previous equality becomes

$$\begin{aligned} \frac{1}{\rho^2} I_0^\sharp[\tan^2\alpha T^2 g - \tan\alpha(1+\tan^2\alpha)Tg] &= -\frac{1}{\rho^2} I_0^\sharp[\partial_\beta^2 g] + I_0^\sharp[-\tan\alpha(1+\tan^2\alpha)Tg + (1+\tan^2\alpha)T^2 g] \\ &= -\frac{1}{\rho^2}\partial_\omega^2 I_0^\sharp g + I_0^\sharp[-\tan\alpha(1+\tan^2\alpha)Tg + (1+\tan^2\alpha)T^2 g]. \end{aligned}$$

Plugging this relation into the right hand side of (49), we obtain

$$\left[(1-\rho^2)\partial_\rho^2 + \left(\frac{1}{\rho} - 3\rho \right)\partial_\rho \right] I_0^\sharp g = -\frac{1}{\rho^2}\partial_\omega^2 I_0^\sharp g + I_0^\sharp[(T^2 + 2\tan\alpha T)g],$$

hence (47) is proved. Equation (48) follows immediately once noticing that

$$D = \frac{1}{\mu} T^2 \mu + 1,$$

thus Theorem 16 is proved. □

The operators $\mathcal{L}$ and $-T^2$. An integration by parts with zero boundary terms (notice that ρ and $1-\rho^2$ both vanish at the ends of $[0,1]$) shows that for all $u, v \in C^\infty(\mathbb{D})$,

$$(\mathcal{L}u, v)_{L^2(\mathbb{D})} = \int_{\mathbb{D}} \left((1-\rho^2)(\partial_\rho u)\overline{(\partial_\rho v)} + \frac{1}{\rho^2}(\partial_\omega u)\overline{\partial_\omega v} \right)\ \rho\ d\rho\ d\omega + (u,v)_{L^2(\mathbb{D})}, \tag{50}$$

in particular $(\mathcal{L}, C^\infty(\mathbb{D})))$ is a symmetric operator when acting on $L^2(\mathbb{D})$. More importantly, it is essentially self-adjoint, in the sense that its operator closure[22] is self-adjoint. In addition, the operator $(-T^2, C^\infty_{\alpha,-,+}(\partial_+ S\mathbb{D}))$ is essentially $L^2_+(\partial_+ S\mathbb{D})$-self-adjoint.

[22] The operator $\overline{\mathcal{L}}$ whose domain is the completion of the pre-Hilbert space $C^\infty(\mathbb{D})$ equipped with the inner product $(f,g) = (f,g)_{L^2} + (\mathcal{L}f, \mathcal{L}g)_{L^2}$, defined as $\overline{\mathcal{L}u} := \lim_{n\to\infty} \mathcal{L}u_n$, where u_n is any sequence in C^∞ converging to u in this topology.

In fact, following notation in [MM21], an orthogonal basis of $L^2_+(\partial_+ S\mathbb{D})$ whose C^∞ span gives $C^\infty_{\alpha,-,+}(\partial_+ S\mathbb{D})$ is given by

$$\psi_{n,k} := \frac{(-1)^n}{4\pi} e^{i(n-2k)(\beta+\alpha)}(e^{i(n+1)\alpha} + (-1)^n e^{-i(n+1)\alpha}), \qquad n \ge 0, \quad k \in \mathbb{Z}, \tag{51}$$

and such that $(-T^2)\psi_{n,k} = (n+1)^2 \psi_{n,k}$ for all n, k.

From these observations, passing to the adjoints in (48), the further intertwining property holds

$$I_0 \circ \mathcal{L} = (-T^2) \circ I_0. \tag{52}$$

Exercise 26. *Prove that for $u, v \in C^\infty(\mathbb{D})$, $(\mathcal{L}u, v)_{L^2(\mathbb{D})} = (u, \mathcal{L}v)_{L^2(\mathbb{D})}$.*

5.4 Backprojecting the joint eigenfunctions of $-T^2$ and $-i\partial_\beta$ - Zernike polynomials

We now focus our attention to $I_0^* \psi_{n,k} = I_0^\sharp \left[\frac{\psi_{n,k}}{\mu}\right]$. Together with the definition of $I_0^\sharp$ and the relations satisfied by the Euclidean footpoint map for all $(\rho e^{i\omega}, \theta) \in S\mathbb{D}$:

$$\begin{aligned} &\beta_-(\rho e^{i\omega}, \theta) + \alpha_-(\rho e^{i\omega}, \theta) + \pi = \theta, \\ &\beta_-\left(\rho e^{i\omega}, \theta\right) = \beta_-(\rho, \theta - \omega) + \omega, \qquad \alpha_-(\rho e^{i\omega}, \theta) = \alpha_-(\rho, \theta - \omega), \end{aligned}$$

we arrive at the expression

$$I_0^\sharp \left[\frac{\psi_{n,k}}{\mu}\right](\rho e^{i\omega}) = e^{i(n-2k)\omega} \frac{1}{2\pi} \int_{\mathbb{S}^1} e^{i(n-2k)\theta} \frac{e^{i(n+1)\alpha_-(\rho,\theta)} + (-1)^n e^{-i(n+1)\alpha_-(\rho,\theta)}}{2\cos\alpha_-(\rho,\theta)}\ d\theta.$$

With the relation $\sin\alpha_-(\rho,\theta) = -\rho\sin\theta$, we may rewrite this as

$$I_0^\sharp \left[\frac{\psi_{n,k}}{\mu}\right](\rho e^{i\omega}) = \frac{e^{i(n-2k)\omega}}{2\pi} \int_{\mathbb{S}^1} e^{i(n-2k)\theta} W_n(-\rho\sin\theta)\ d\theta, \tag{53}$$

where we have defined

$$W_n(\sin\alpha) := \frac{e^{i(n+1)\alpha} + (-1)^n e^{-i(n+1)\alpha}}{2\cos\alpha}. \tag{54}$$

The functions W_n are related to the Chebychev polynomials of the second kind U_n, specifically through the relation $W_n(t) = i^n U_n(t)$. In particular, it is immediate to check the 2-step recursion relation and initial conditions

$$W_{n+1}(t) = 2itW_n(t) + W_{n-1}(t), \qquad W_0(t) = 1, \qquad W_1(t) = 2it.$$

By induction, the top-degree term of W_n is $(2it)^n$. Fixing $n \ge 0$, we now split the calculation into two cases:

Case $k<0$ or $k>n$. In light of (53), since W_n is a polynomial of degree n, then $W_n(-\rho\sin\theta)$ is a trigonometric polynomial of degree n in $e^{i\theta}$. In particular, if $k<0$ or $k>n$, then $|n-2k|>n$ and thus the right hand side of (53) is identically zero. In short, we deduce

$$I_0^\sharp\left[\frac{\psi_{n,k}}{\mu}\right]=0, \qquad n\ge 0, \qquad k<0 \text{ or } k>n.$$

Case $0\le k\le n$. For the remaining cases, we then define $Z_{n,k}:=I_0^\sharp\left[\frac{\psi_{n,k}}{\cos\alpha}\right]$, and for the sake of self-containment, we now show that the functions $\{Z_{n,k}\}_{n\ge 0,\ 0\le k\le n}$ so constructed are the Zernike basis in the convention of [KB04], by showing that they satisfy Cauchy-Riemann systems and take the same boundary values.

Lemma 17. *The functions $\{Z_{n,k}\}_{n\ge 0,\ 0\le k\le n}$ satisfy the following properties: For all $n\ge 0$*

$$\partial_{\bar z} Z_{n,0}=0, \qquad \partial_z Z_{n,k}+\partial_{\bar z}Z_{n,k+1}=0 \qquad (0\le k\le n-1), \qquad \partial_z Z_{n,n}=0, \tag{55}$$

$$Z_{n,k}(e^{i\omega})=(-1)^k e^{i(n-2k)\omega}, \qquad 0\le k\le n, \quad \omega\in\mathbb{S}^1. \tag{56}$$

Proof. Using the relation $W_n(-t)=(-1)^nW_n(t)$, we arrive at the expression

$$\begin{aligned} Z_{n,k}(\rho e^{i\omega}) &= e^{i(n-2k)\omega}\frac{(-1)^n}{2\pi}\int_{\mathbb{S}^1} e^{i(n-2k)\theta}W_n(\rho\sin\theta)\ d\theta \\ &= \frac{(-1)^n}{2\pi}\int_{\mathbb{S}^1} e^{i(n-2k)\theta}W_n(\rho\sin(\theta-\omega))\ d\theta. \end{aligned} \tag{57}$$

With $\partial_z=\frac{e^{-i\omega}}{2}(\partial_\rho-\frac{i}{\rho}\partial_\omega)$ and $\partial_{\bar z}=\frac{e^{i\omega}}{2}(\partial_\rho+\frac{i}{\rho}\partial_\omega)$, we compute

$$\partial_z(\rho\sin(\theta-\omega))=i\frac{e^{-i\theta}}{2}, \qquad \partial_{\bar z}(\rho\sin(\theta-\omega))=-i\frac{e^{i\theta}}{2}.$$

Plugging these into (57) immediately implies

$$\partial_z Z_{n,k}+\partial_{\bar z}Z_{n,k+1}=0, \qquad 0\le k\le n-1. \tag{58}$$

In addition, we compute

$$\begin{aligned} Z_{n,0}(\rho e^{i\omega}) &= e^{in\omega}\frac{(-1)^n}{2\pi}\int_{\mathbb{S}^1} e^{in\theta}W_n(\rho\sin\theta)\ d\theta \\ &= e^{in\omega}\frac{(-1)^n}{2\pi}\int_{\mathbb{S}^1} e^{in\theta}(2i\rho\sin\theta)^n\ d\theta \\ &= \rho^n e^{in\omega}\frac{(-1)^n}{2\pi}\int_{\mathbb{S}^1} e^{in\theta}(2i\sin\theta)^n\ d\theta \end{aligned}$$

where the second equality comes from the fact that the lower-order terms of $W_n(\rho\sin\theta)$ have no harmonic content along $e^{in\theta}$. Finally, the constant is

$$\int_{\mathbb{S}^1} e^{in\theta}(e^{i\theta}-e^{-i\theta})^n\ d\theta=\int_{\mathbb{S}^1}(e^{2i\theta}-1)^n\ d\theta=2\pi(-1)^n.$$

In short, $Z_{n,0} = \rho^n e^{in\omega} = z^n$. This also implies $\partial_{\overline{z}} Z_{n,0} = 0$ and since we have $Z_{n,n} = (-1)^n \overline{Z_{n,0}} = (-1)^n \overline{z}^n$, we deduce that $\partial_z Z_{n,n} = 0$.

To prove the boundary condition, using that $Z_{n,k}(\rho e^{i\omega}) = e^{i(n-2k)\omega} Z_{n,k}(\rho)$, it is enough to show that $Z_{n,k}(1) = (-1)^k$ for every $n \ge 0$ and $0 \le k \le n$. That this is true for $k = 0$ and $k = n$ follows from the expressions just computed, and the general claim follows by induction on n once the following equality is satisfied:

$$Z_{n,k}(1) = Z_{n-2,k-1}(1) - Z_{n-1,k-1}(1) + Z_{n-1,k}(1). \tag{59}$$

To prove (59), it suffices to input the recursion $W_n(\sin\theta) = 2i\sin\theta W_{n-1}(\sin\theta) + W_{n-2}(\sin\theta)$ into the expression (57), and to evaluate it at $\rho e^{i\omega} = 1$. $\square$

From Lemma 17, we see that the family so defined satisfies the characterization (b) of [KB04, Theorem 1] of the Zernike polynomials. One may see that this characterization defines the same family due the following facts: for $n \ge 0$ and $k = 0$, the functions $Z_{n,k}$ in both sets agree; by induction on $k > 0$, in both sets of functions, $Z_{n,k}$ satisfies a $\partial_{\overline{z}}$ equation with same right-hand side and same boundary condition, for which a solution is unique if it exists.

Let us then give a few useful properties of these polynomials:

- The following characterization is proved in [KB04, Theorem 1]:

$$Z_{n,k}(z, \overline{z}) = \frac{1}{k!} \frac{\partial^k}{\partial z^k} \left[z^n \left(\frac{1}{z} - \overline{z} \right)^k \right], \quad n \ge 0, \quad 0 \le k \le n. \tag{60}$$

- The family $\{Z_{n,k}\}_{n\ge 0, 0\le k\le n}$ is orthogonal on $L^2(\mathbb{D})$. Indeed, they are the eigenfunction of the pair of self-adjoint operators $(\mathcal{L}, -\partial_\omega^2)$ (as densely defined on $C^\infty(\mathbb{D})$), since we have

$$(\mathcal{L}, -\partial_\omega^2) Z_{n,k} = (\mathcal{L}, -\partial_\omega^2) I_0^* \psi_{n,k} = I_0^*(-T^2, -\partial_\beta^2)\psi_{n,k} = ((n+1)^2, (n-2k)^2) Z_{n,k},$$

  and the map $(n,k) \mapsto ((n+1)^2, (n-2k)^2)$ is injective.

- Their completeness in $L^2(\mathbb{D})$ follows again from the Weierstrass approximation theorem.

We finally show

Lemma 18.

$$\|Z_{n,k}\|^2 = \frac{\pi}{n+1}, \qquad n \ge 0, \qquad 0 \le k \le n. \tag{61}$$

A functional-analytic proof is given in [Mon20, Appendix]. We give here a proof in the spirit of recurrence relations for orthogonal polynomials.

Proof of Lemma 18. As a quick consequence of (60), the following relation holds

$$(n+1) Z_{n,k} = -\overline{\partial}(Z_{n+1,k+1} + Z_{n-1,k}), \qquad n \ge 0,\ 0 \le k \le n. \tag{62}$$

Multiplying (62) by $\overline{Z_{n,k}}$ and integrating, we arrive at the relation

$$(n+1)\|Z_{n,k}\|^2 = -\int_{\mathbb{D}} (\overline{\partial} Z_{n+1,k+1})\overline{Z_{n,k}} + \int_{\mathbb{D}} (\overline{\partial} Z_{n-1,k})\overline{Z_{n,k}}.$$

The last term is zero because $\overline{\partial} Z_{n-1,k}$ is of degree $n-2$ and $Z_{n,k}$ is orthogonal to any polynomial of degree $\ge n-1$. On to the first term,

$$\begin{aligned} \int_{\mathbb{D}} (\overline{\partial} Z_{n+1,k+1})\overline{Z_{n,k}} &= \int_{\mathbb{D}} \overline{\partial}(Z_{n+1,k+1}\overline{Z_{n,k}}) - \int_{\mathbb{D}} Z_{n+1,k+1}\overline{\partial Z_{n,k}}, \\ &= -\frac{1}{2i}\int_{\partial\mathbb{D}} Z_{n+1,k+1}\overline{Z_{n,k}}\ dz - \int_{\mathbb{D}} Z_{n+1,k+1}\overline{\partial Z_{n,k}}. \end{aligned}$$

The rightmost term is again zero by consideration of degree, while the boundary term is computed using (56), to wit

$$\int_{\mathbb{D}} (\overline{\partial} Z_{n+1,k+1}\overline{Z_{n,k}})\ dz = \frac{-1}{2i}\int_{\mathbb{S}^1} (-1)^{k+1} e^{i(n+1-2(k+1))\beta} e^{-i(n-2k)\beta} (-1)^k i e^{i\beta}\ d\beta = \frac{1}{2}\int_{\mathbb{S}^1} d\beta = \pi,$$

hence (61) is proved. □

Exercise 27. *Prove* (62) *using* (60)*.*

5.5 SVD of I_0 and mapping properties

We now conclude regarding the SVD of I_0 using the method of intertwining differential operators. This SVD has been known for quite some time, see e.g. [Cor64, Lou84], and the idea to use intertwining differential operators for such derivations can be found e.g. in [Maa91], though they are usually written there for each polar harmonic number separately.

Equation (48) allows to avoid this separation by harmonics. Below, the "hat" notation stands for vector normalization in their respective spaces.

Theorem 19. *The Singular Value Decomposition of* $I_0\colon L^2(\mathbb{D}) \to L^2(\partial_+ S\mathbb{D}, d\Sigma^2)$ *is given by*

$$(\widehat{Z_{n,k}}, \widehat{\psi_{n,k}}, a_{n,k})_{n\ge 0, 0\le k\le n}, \qquad a_{n,k} := \frac{\sqrt{4\pi}}{\sqrt{n+1}}. \tag{63}$$

Proof. We obviously have $(-T^2)\psi_{n,k} = (n+1)^2\psi_{n,k}$ and $-i\partial_\beta \psi_{n,k} = (n-2k)\psi_{n,k}$, which by self-adjointness on $L^2(\partial_+ S\mathbb{D}, d\Sigma^2)$ of the two operators applied, makes $\psi_{n,k}$ and orthogonal system. In addition, an immediate computation gives

$$\|\psi_{n,k}\|^2_{L^2(\partial_+ S\mathbb{D})} = \frac{1}{4}, \qquad n\ge 0, \qquad k\in\mathbb{Z}.$$

In addition we have, as explained in [MM21] $I_0^*\psi_{n,k} = 0$ for $k<0$ or $k>n$, and for $0\le k\le n$, we define $Z_{n,k} := I_0^*\psi_{n,k}$. By Theorem 16, we compute

$$\begin{aligned} \mathcal{L} Z_{n,k} &= \mathcal{L} I_0^*\psi_{n,k} = I_0^*(-T^2)\psi_{n,k} = (n+1)^2 Z_{n,k} \\ -i\partial_\omega Z_{n,k} &= (n-2k)Z_{n,k}, \end{aligned}$$

which immediately makes them an orthogonal system in $L^2(\mathbb{D})$. This gives us orthogonal systems associated with I_0 and I_0^* and to compute the singular values, it suffices to normalize all vectors. By definition we have

$$I_0^* \ \widehat{\psi_{n,k}} = a_{n,k} \ \widehat{Z_{n,k}}, \qquad a_{n,k} := \frac{\|Z_{n,k}\|_{L^2(\mathbb{D})}}{\|\psi_{n,k}\|_{L^2(\partial_+ S\mathbb{D})}} = 2\|Z_{n,k}\|_{L^2(\mathbb{D})} \stackrel{(61)}{=} \frac{\sqrt{4\pi}}{\sqrt{n+1}},$$

hence the result. □

The following statement follows directly. Although it is unclear whether it appears explicitly in the literature, the ingredients for the proof were known since Zernike's seminal paper [Zer34].

Theorem 20. *The following relation holds:*

$$\mathcal{L}(I_0^* I_0)^2 f = (4\pi)^2 f, \qquad f \in C^\infty(\mathbb{D}).$$

Proof. The proof is seen at the level of the spectral decomposition, since we have for every $n \ge 0$ and $0 \le k \le n$,

$$I_0^* I_0 Z_{n,k} = \frac{4\pi}{n+1} Z_{n,k}, \qquad \text{and} \qquad \mathcal{L} Z_{n,k} = (n+1)^2 Z_{n,k}.$$

□

Mapping properties of $I_0^* I_0$. For $s \in \mathbb{R}$, let us define the scale of Hilbert spaces

$$\begin{aligned} \widetilde{H}^s(\mathbb{D}) &= \left\{ f = \sum_{n=0}^\infty \sum_{k=0}^n f_{n,k} \widehat{Z_{n,k}}, \qquad \sum_{n=0}^\infty (n+1)^{2s} \sum_{k=0}^n |f_{n,k}|^2 < \infty \right\} \\ &= \left\{ f \in L^2(\mathbb{D}), \ \mathcal{L}^{s/2} f \in L^2(\mathbb{D}) \right\}, \end{aligned} \tag{64}$$

with continuous, in fact compact, injections $\widetilde{H}^s \subset \widetilde{H}^t$ for $s > t$. An important property of the scale $\{\widetilde{H}^s(\mathbb{D})\}_s$ is the following:

Theorem 21.

$$\bigcap_{s \in \mathbb{R}} \widetilde{H}^s(\mathbb{D}) = C^\infty(\mathbb{D})$$

Proof. The inclusion $\supset$ is clear, since a smooth function f is such that for all $n \ge 0$, $\mathcal{L}^n f \in L^2(\mathbb{D})$. The proof of the inclusion $\subset$ is based on the next two lemmas, proved in [Mon20, Appendix].

Lemma 22. *For all $\alpha > 3/2$, we have the continuous injection $\widetilde{H}^\alpha(\mathbb{D}) \to C(\mathbb{D})$.*

Lemma 23. *There exists $\ell > 0$ such that for every $\alpha \ge \ell$, the operators*

$$\partial \colon \widetilde{H}^\alpha(\mathbb{D}) \to \widetilde{H}^{\alpha-\ell}(\mathbb{D}) \qquad \textit{and} \qquad \overline{\partial} \colon \widetilde{H}^\alpha(\mathbb{D}) \to \widetilde{H}^{\alpha-\ell}(\mathbb{D})$$

are bounded. The index ℓ can be chosen as $2 + \varepsilon$ for every $\varepsilon > 0$.

To prove the inclusion $\subset$, it is enough to show that if $f \in \cap_{s\ge 0}\widetilde{H}^s(\mathbb{D})$, then for any $p,q\ge 0$, $\partial^p\overline{\partial}^q f \in C(\mathbb{D})$. With ℓ a constant as in Lemma 23, since $f \in \widetilde{H}^{(p+q)\ell+3}(\mathbb{D})$, repeated use of Lemma 23 gives that $\partial^p\overline{\partial}^q f \in \widetilde{H}^3(\mathbb{D})$, and by Lemma 22, this implies that $\partial^p\overline{\partial}^q f \in C(\mathbb{D})$. Hence the result. $\square$

Moreover, it is immediate to establish the property, for all $s \ge 0$

$$\|I_0^* I_0 f\|_{\widetilde{H}^{s+1}} = 4\pi \|f\|_{\widetilde{H}^s}, \qquad \forall f \in \widetilde{H}^s, \tag{65}$$

which is both a continuity and stability estimate.

Mapping properties of I_0. We also fully understand the mapping properties of $I_0\colon L^2(M) \to L^2_+(\partial_+ S\mathbb{D})$: if $f = \sum_{n\ge 0}\sum_{k=0}^n f_{n,k}\widehat{Z_{n,k}}$, then

$$I_0 f = \sum_{n\ge 0}\sum_{k=0}^n f_{n,k}\frac{\sqrt{4\pi}}{\sqrt{n+1}}\widehat{\psi_{n,k}}.$$

The intertwining property (52) suggests that I_0 translates regularity w.r.t. to $\mathcal{L}$ into regularity as described by the vector field T only. This justifies that, similar to the case of the Radon transform where the Sobolev scale on $\mathcal{Z}$ was only describing regularity w.r.t. d/ds, one should construct an anisotropic Sobolev scale on $\partial_+ S\mathbb{D}$ describing regularity w.r.t. T. Indeed let's define

$$\begin{aligned} H^s_{T,+}(\partial_+ S\mathbb{D}) &= \left\{ g = \sum_{n=0}^\infty \sum_{k\in\mathbb{Z}} g_{n,k}\widehat{\psi_{n,k}}, \quad \sum_{n=0}^\infty (n+1)^{2s}\sum_{k\in\mathbb{Z}} |g_{n,k}|^2 < \infty \right\} \\ &= \left\{ g \in L^2_+(\partial_+ S\mathbb{D}), \quad (-T^2)^{s/2} g \in L^2_+(\partial_+ S\mathbb{D}) \right\}, \end{aligned}$$

in which the following identity is immediate:

$$\|If\|_{H^{s+1/2}_{T,+}} = \sqrt{4\pi}\|f\|_{\widetilde{H}^s}, \qquad \forall f \in \widetilde{H}^s. \tag{66}$$

Exercise 28. *What do d/ds and $\partial/\partial\theta$ look like in fan-coordinates ?*

Exercise 29. *Show (65) and (66).*

Unlike $I_0^* I_0$ which is surjective from $\widetilde{H}^s$ to $\widetilde{H}^{s+1}$, and although (66) holds, this does not say that I_0 is surjective, as equality (66) is indexed by f and says nothing as to whether If exhausts the codomain.

And indeed, functions on the range of I_0 are seen to be linear combinations of $\psi_{n,k}$ for $n \in \mathbb{N}_0$ and $0 \le k \le 0$. In particular, for every $n \in \mathbb{N}_0$, k misses the set $\mathbb{Z}\backslash\{0,\dots,n\}$.

In [Mon20, Appendix], one defines an operator $C_-\colon L^2(\partial_+ S\mathbb{D}) \to L^2(\partial_+ S\mathbb{D})$ that is $H^s_{T,+} \to H^s_{T,+}$-continuous for every s, that acts diagonally on each $\psi_{n,k}$, and such that $C_-\psi_{n,k} = 0$ if and only if $n \in \mathbb{N}_0$ and $0 \le k \le n$. This allow the final mapping refinement:

$$I_0(\widetilde{H}^s(\mathbb{D})) = H^{s+\frac{1}{2}}_{T,+}(S\mathbb{D}) \cap \ker C_-. \tag{67}$$

Going further. . .

- In the case of the Euclidean X-ray transform on $\mathbb{D}$, we have shown that one could construct special Sobolev scales modeled after differential operators, which in turned sharply captured the mapping properties of I_0. A natural question to ask, in the context of the geodesic X-ray transform on simple Riemannian surfaces, is:

  Given (M, g) a simple Riemannian surface and I_0 the geodesic X-ray transform defined on it, how to design appropriate Sobolev scales that sharply capture the mapping properties of I_0 ? Does the functional link between $I_0^* I_0$ and degenerate elliptic operators persist on other surfaces ?

  A positive answer to the last question in the case of simple geodesic disks in constant curvature spaces was given in [Mon20], though the borderline cases remain to be established.

- The Sobolev scale introduced in (64) is adapted to other operators such as weighted X-ray transforms on the Euclidean disk (or constant curvature disks). Denoting I_Θ the attenuated X-ray transform in [MNP21] with attenuation $\Theta \in C_c^\infty(\mathbb{D}^{int})$, it is shown that $I_\Theta^* I_\Theta$ is a relatively compact perturbation of $I_0^* I_0$ on the scale (64), and this allows to sharply capture the mapping properties of these operators as well.

- Even in the case of the unit disk, there is a way to slightly perturb the story told above, where the construction of a Sobolev scale remains possible yet it does not isometrically capture the mapping properties of the X-ray transform[23]. The main results are in [MMZ22].

  On $\mathbb{D}$, let $d := 1 - \rho^2$, a boundary defining function. For $\gamma > -1$, one may show that the "singularly weighted" X-ray transform $I_0 d^\gamma \colon L^2(\mathbb{D}, d^\gamma) \to L^2(\partial_+ S\mathbb{D}, \mu^{-2\gamma})$ is bounded, with adjoint $I_0^* \mu^{-2\gamma}$. (note here that the notations may slightly differ from [MMZ22] since our definition of I_0^* also contains another μ^{-1} in it). It is then found that there exist natural self-adjoint operators $\mathcal{L}_\gamma$ and $\mathcal{T}_\gamma$, whose spectral decomposition is well-understood, and with the intertwining relations

  $$I_0^* \mu^{-2\gamma} \circ \mathcal{T}_\gamma = \mathcal{L}_\gamma \circ I_0^* \mu^{-2\gamma}, \qquad I_0^* \mu^{-2\gamma} \circ \partial_\beta = \partial_\omega \circ I_0^* \mu^{-2\gamma},$$

  Then the method of intertwining differential operators applies, and we can compute the SVD of $I_0^* \mu^{-2\gamma} I_0 d^\gamma$, which in passing can be proved to be an isomorphism of $C^\infty(\mathbb{D})$.

  However unless $\gamma = 0$, the eigendecomposition of $I_0^* \mu^{-2\gamma} I_0 d^\gamma$ is such that its eigenvalues depend on n AND on k (the same spectral parameters as in the case $\gamma = 0$), while those of $\mathcal{L}_\gamma$ depend on n only, and as such one can no longer write a relation between the two operators, which in turn would capture the mapping properties of $I_0^* \mu^{-2\gamma} I_0 d^\gamma$ using Sobolev spaces defined after $\mathcal{L}_\gamma$. One could wonder whether another operator than $\mathcal{L}_\gamma$ would do, although given everything being so explicit (spectral decomposition, spaces, etc.), it is hard to imagine.

[23] in the sense that the smoothing exponent will not match with the ill-posedness exponent.

- The method of intertwining differential operators seems to require lots of symmetries to even be applicable, which might limit its applicability in the context of general manifolds. On the other hand, it might be the case that this method could apply to symmetric spaces as covered in Helgason's book [Hel10], which was largely written in cases without boundary.

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