Introduction to Inverse Problems and Integral Geometry
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1 Introduction

Though we will later focus on inverse problems in integral geometry, these problems sit inside the larger field of inverse problems, which prescribes an agenda of questions that one may address. Some examples may then not be directly related to integral geometry, though they are part of the inverse problems folklore (some of them still open to this day).

Reading material: [Bal12, Chapter 1], [Ilm17, §1]

1.1 What is an inverse problem ?

Given a continuous map $\mathcal{M} : \mathcal{X} \to \mathcal{Y}$ and $y \in \mathcal{Y}$, find $x \in \mathcal{X}$ such that $\mathcal{M}(x) = y$.

- $\mathcal{M}$ can be linear or non-linear.
- $\mathcal{X}$ and $\mathcal{Y}$ can be finite-dimensional normed spaces or, most often, infinite-dimensional topological vector spaces (Banach, Hilbert, Fréchet spaces of functions and distributions) or topological manifolds.
- The spaces above often reflect a notion of **smoothness**, typically embodied by a scale of Hilbert or Banach spaces (e.g. Sobolev spaces), whose intersection gives a Fréchet space (e.g. smooth functions, rapidly-decaying functions or both). A **scale** of spaces is a family of Banach or Hilbert spaces $\{(H^s, \|\cdot\|_s\}_{s\in\mathbb{N}_0}$ with continuous injections $H^s \hookrightarrow H^t$, $s \ge t$, and the intersection may be given a Fréchet topology defined by the countable family of seminorms $\|\cdot\|_s, s \in \mathbb{N}_0$. Important examples include Sobolev spaces ($f \in H^k(\mathbb{R})$ iff $\partial^\alpha f \in L^2$ for all $0 \le \alpha \le k$, with norm $\|f\|_{H^k}^2 = \sum_{\alpha=0}^k \|\partial^\alpha f\|_{L^2}^2$) and the classical $C^k(\mathbb{R})$ Banach spaces (equipped with the sup norm $\|f\|_{C^k} := \max_{0\le\alpha\le k} \sup_{x\in\mathbb{R}} |\partial^\alpha f(x)|$), with intersection space $C^\infty(\mathbb{R})$. Another way to quantify smoothness is by means of decay in the Fourier domain, which may motivate other scales of spaces of the form $B^k = \{f \in L^2(\mathbb{R}); (1+x^2)^{k/2} f \in L^2(\mathbb{R})\}$, $k \ge 0$.
- $\mathcal{M}$ may be not injective and not surjective. What, then, does it mean, to 'invert' $\mathcal{M}$?
- It could be that the problem may be formulated as "recover x from $\mathcal{M}(x)$", and part of the job is to find a 'good' pair of spaces as well.

Some examples:

- Finite-dimensional, linear inverse problems.
- Recovering $f\colon \mathbb{R} \to \mathbb{R}$ from its antiderivative $F(x) = \int_0^x f(t)\ dt$.
- Recovering f from its Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\ dx$, or its Hilbert transform $Hf(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{x-y} f(y)\ dy$.

- Inverse diffusion: to recover the initial temperature distribution $u_0(x)$ from the observation of the temperature distribution at a later time $u_T(x) = u(x,T)$, where $u(x,t)$ solves the heat equation

$$\begin{aligned} \partial_t u = \partial_x^2 u, \qquad x \in \mathbb{R}, \quad t > 0 \qquad &\text{(heat equation)} \\ u(x,0) = u_0(x). \qquad &\text{(initial condition)} \end{aligned}$$

- X-ray/Radon transform (see below).

- Calderon's problem, boundary rigidity, inverse spectral problem.

1.2 The inverse problems agenda

Two practically motivated philosophical points may help approach what follows:

- The forward operator $\mathcal{M}$ is generally 'smoothing' and therefore, undoing that process will require 'unsmoothing' the data, a process which is in itself ill-conditioned (or unstable) in the sense that $\mathcal{M}^{-1} \colon \mathcal{Y} \to \mathcal{X}$ may not be continuous in general.

- The data y may be corrupted by noise, in the sense that we want to find $x \in \mathcal{X}$ measuring $\mathcal{M}(x) + \eta$, where η models a noise realization which we have little control over. Two issues naturally arise out of this: first, the measurement $\mathcal{M}(x) + \eta$ might no longer live in $\mathcal{Y}$ (depending on how smoothing the problem is, $\mathcal{M}(x)$ may be smoother than η, so $\mathcal{M}(x) + \eta$ lives in the space where η lives, not where $\mathcal{M}(x)$ lives); second, $\mathcal{M}(x) + \eta$ might not even be in the range of $\mathcal{M}$!

With that in mind, here are some of the important questions to be addressed:

Injectivity: Does $\mathcal{M}(x)$ characterize x uniquely (equivalently, does $\mathcal{M}(x_1) = \mathcal{M}(x_2)$ imply $x_1 = x_2$) ? If not, can one describe obvious obstructions to injectivity (for example, is there a "gauge" group action $\mathcal{G} \times \mathcal{X} \to \mathcal{X}$ such that $\mathcal{M}(x_1) = \mathcal{M}(x_2)$ if and only if $x_1 = g \cdot x_2$ for some $g \in \mathcal{G}$) ? If yes, the lack of injectivity is well-understood, and one could in principle replace the initial, non-injective operator $\mathcal{M} \colon \mathcal{X} \to \mathcal{Y}$ by the injective one $\mathcal{M} \colon \mathcal{X}/\mathcal{G} \to \mathcal{Y}$.

Stability: Suppose injectivity has been settled. Stability is a quantification of how 'well-behaved' the inversion process will be.

One way to think of stability is to ask what is the form of the modulus of continuity of $\mathcal{M}^{-1} : \mathcal{Y} \to \mathcal{X}$, if any ? i.e., for which function $\omega : [0,\infty) \to [0,\infty)$ with $\lim_{x\to 0} \omega(x) = 0$ do we have[1]

$$\|\mathcal{M}^{-1}(y_1) - \mathcal{M}^{-1}(y_2)\|_{\mathcal{X}} \lesssim \omega(\|y_1 - y_2\|_{\mathcal{Y}}).$$

[1]We will write $a \lesssim b$ if there is a constant C such that $a \le Cb$. Unless important, we will not keep track of constants.

Since $\mathcal{M}^{-1}$ might not even make sense, we prefer writing

$$\|x_1 - x_2\|_{\mathcal{X}} \lesssim \omega(\|\mathcal{M}(x_1) - \mathcal{M}(x_2)\|_{\mathcal{Y}}). \tag{1}$$

Equation (1) is a *stability estimate*, which quantifies how error on data (measured with $\|\cdot\|_{\mathcal{Y}}$) translates into error on the reconstruction (measured with $\|\cdot\|_{\mathcal{X}}$). It helps answer the question: suppose I want a reconstruction error no greater than ε, what precision on my measurements do I need ?

The problem is then said *Lipschitz-stable in* $(\mathcal{X}, \mathcal{Y})$ if one has equality (1) with $\omega(x) = x$, the best-case scenario[2]; *Hölder-stable in* $(\mathcal{X}, \mathcal{Y})$ if $\omega(x) = x^\alpha$ for some $0 < \alpha < 1$; worse moduli of continuity include $\omega(x) = \frac{1}{|\log x|}$ (*log-stable*, or exponentially ill-posed).

A problem may be made Lipschitz-stable for some specific choice of norm, in spite of the fact that it is objectively badly-behaved. But then, this Lipschitz estimate is probably practically unusable because the space $\mathcal{Y}$ is too small for the noise to live in it. Although we mentioned above that there is some leeway in choosing the spaces $\mathcal{X}$ and $\mathcal{Y}$, a first constraint is to use a space $\mathcal{Y}$ where the noise lives.

Another way to define stability is with respect to Hilbert scales $\mathcal{X}_k$ and $\mathcal{Y}_k$ such that $\mathcal{M}\colon \mathcal{X}_k \to \mathcal{Y}_k$ is continuous for all k. Then the inverse problem is

- *well-posed* if $\|x - x'\|_k \lesssim \|\mathcal{M}(x) - \mathcal{M}(x')\|_k$ for all k
- *mildly ill-posed of order* $\alpha > 0$ if there is $\alpha > 0$ such that $\|x - x'\|_k \lesssim \|\mathcal{M}(x) - \mathcal{M}(x')\|_{k+\alpha}$ for all k. One then seeks the smallest α and calls it the *order of ill-posedness of* $\mathcal{M}$ (it depends on the choice of scales)
- *severely ill-posed* otherwise.

When the grading of the Hilbert scales at play describes order of differentiability, the α above quantifies by how many derivatives the operator $\mathcal{M}$ is smoothing (as a result, reconstructing x will involve differentiating the data α times). A severely ill-posed problem typically corresponds to an operator which is smoothing by an infinite degree.

What is the link between the above two notions ? When the second one is well-understood, one can cook up appropriate moduli of continuity for the inverse, provided that one adds a prior smoothness assumption on the unknown x (see Exercise 3).

Range characterization: The operator $\mathcal{M} : \mathcal{X} \to \mathcal{Y}$ is most likely not surjective, so how does $\mathcal{M}(\mathcal{X})$ sit inside $\mathcal{Y}$? Given $y \in \mathcal{Y}$, can one find "consistency conditions" which imply that y is in the range of $\mathcal{M}$? If $\mathcal{M}$ is linear, then $\mathcal{M}(\mathcal{X})$ is a linear subspace of $\mathcal{Y}$; can one easily describe the supplementary (sometimes, orthocomplement) of $\mathcal{M}(\mathcal{X})$?

For example,

Consider the operator $A\colon \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ given by $A\mathbf{u} = \left(\frac{1-(-1)^n}{n} u_n\right)_n$, $\mathbf{u} = (u_n)_n$. Equip $\ell^2(\mathbb{N}_0)$ with the orthonormal family $\{\mathbf{e}_n\}_{n\ge 0}$ where for $n \ge 0$, $\mathbf{e}_n = \{\delta_{jn}\}_{j\ge 0}$.

[2] Recall from one-variable calculus: a function with superlinear modulus of continuity is constant.

A is not surjective for two fundamentally different reasons: any basis vector $\mathbf{e}_{2k}$ is not achieved in the range of A, and if we were to restrict A to the span $\langle \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5, \dots \rangle$, where it becomes injective, it is still not surjective because elements in the range have faster decay than ℓ^2. In particular, the sequence $\mathbf{u} = \left\{ \frac{1+(-1)^n}{n} \right\}_n \in \ell^2(\mathbb{N}_0)$ seems to have a preimage by A, but that preimage does not belong to $\ell^2(\mathbb{N}_0)$.

Reconstruction: What are the ways that we can recover x from $\mathcal{M}(x)$? (explicit reconstruction formulas, Fredholm equations, regularized inversions, Markov Chain Monte Carlo)

Partial Data problems: What if we only have partial knowledge of $\mathcal{M}(x)$ (e.g. discrete samples, or a restriction of the full data) ? How are injectivity, stability and reconstruction impacted ?

Parameter dependence: Similarly to the previous question: how do the answers to injectivity, stability and reconstruction depend on some parameter in the system considered ? Examples:

- Injectivity and stability of the geodesic X-ray transform on a Riemannian surface depend on whether that surface has conjugate points.
- Injectivity of the attenuated X-ray transform over vector fields degenerates as the attenuation vanishes.
- In an inverse wave problem (TAT/PAT), injectivity and stability depend on the observation time.

Practical questions: What is the nature of the noise in the measurement ? (in what space does it live ?) how to use the stability estimate to understand how errors will magnify ? How much do we have to regularize the inversion in order to obtain a meaningful reconstruction ?

1.3 Some prototypes

- Finite-dimensional, linear inverse problems. Singular Value Decomposition. Recall that for a linear operator $A\colon \mathbb{C}^p \to \mathbb{C}^q$, setting $r = \min(p, q)$, there exists orthonormal bases $(u_1, \dots, u_p)$ and $(v_1, \dots, v_q)$ and non-negative numbers $\sigma_1, \dots, \sigma_r$ such that

  $$Au_j = \sigma_j v_j, \quad A^* v_j = \sigma_j u_j, \qquad 1 \le j \le r.$$

  If $p > q$, then we also have $Au_j = 0$ for $q < j \le p$ and if $q > p$, we have $A^* v_j = 0$ for $p \le j < q$. The u_j's are the eigenvectors of $A^*A\colon \mathbb{C}^p \to \mathbb{C}^p$ (a symmetric operator), the v_j's are the eigenvectors of $AA^*\colon \mathbb{C}^q \to \mathbb{C}^q$, and σ_j^2 are the eigenvalues of either operator.

  In the case where $p = q$ and all singular values are non-zero and arranged in decreasing order $\sigma_1 \ge \dots \ge \sigma_p$, then we have

  $$\sigma_p \|x\| \le \|Ax\| \le \sigma_1 \|x\|, \qquad \forall x \in \mathbb{C}^p, \tag{2}$$

  which gives us both continuity and stability constants.

- Infinite-dimensional linear inverse problems involving compact[3] operators. Suppose A is now a linear, bounded operator between two Hilbert spaces $A\colon \mathcal{H}_1 \to \mathcal{H}_2$. Can we use the SVD picture again ? Well, not always. The operator $A^*A\colon \mathcal{H}_1 \to \mathcal{H}_1$ will be bounded, self-adjoint, but its spectrum may not be discrete[4]. However, if A^*A is *compact*, then indeed, we can find two Hilbert orthonormal bases u_n, v_n of $\mathcal{H}_1, \mathcal{H}_2$ and a decreasing sequence of non-negative numbers σ_j such that

$$Au_j = \sigma_j v_j, \qquad A^* v_j = \sigma_j u_j, \qquad j \ge 0.$$

  This is the spectral theorem for self-adjoint, compact operators. Moreover, since A^*A is compact, the sequence σ_j necessarily decreases to zero, so $\inf_j \sigma_j = 0$ and there will be no room to write a stability estimate as in (2). Having an SVD is however extremely relevant to understand stability, inversion and regularization purposes.

- Recover $f \in L^2([0, 2\pi])$ from its Fourier coefficients

$$a_n[f] = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(\theta)\ d\theta, \quad n \in \mathbb{Z}.$$

- Suppose $f \in L^2([0, 2\pi])$ satisfies $\int_0^{2\pi} f(\theta)\ d\theta = 0$. Recover f from the functional

$$F(\theta) = \int_0^\theta f(\theta')\ d\theta', \qquad \theta \in [0, 2\pi].$$

- Consider the measurement operator as follows: Given $f \in L^2([0, \pi])$, Let $\mathcal{M}(f) = u|_{t=T}$, where $u(x, t)$ solves the heat equation

$$\partial_t u = \partial_{xx} u \qquad (0, \pi) \times (0, T), \qquad u|_{t=0} = f,$$

  with Neuman boundary conditions $\partial_x u(0, t) = \partial_x u(\pi, t) = 0$.

  1. Expand f and u as Fourier cosine series to make appear an explicit form of the measurement operator. In the Fourier cosine series identification $f \leftrightarrow \{a_n\}_{n\ge 0}$ such that $f(x) = \sum_{n=0}^\infty a_n \cos(nx)$, the operator $\mathcal{M}$ is diagonal and its action looks like $a_n \mapsto a_n e^{-n^2 T}$. In particular, the operator $\mathcal{M}$ (or, rather $\mathcal{F}\mathcal{M}\mathcal{F}^{-1}$ with $\mathcal{F}$ the cosine series transform) is bounded and injective from $\ell^2(\mathbb{N}_0)$ into itself.
  2. On the other hand, the inverse is not $\ell^2 \to \ell^2$ continuous, nor is it $h^p \to \ell^2$ bounded for any $p \ge 0$ (study the ratio $\frac{\|f\|_{\ell^2}}{\|\mathcal{M}(f)\|_{h^p}}$ with $f(x) = \cos(nx)$). This is an example of an exponentially ill-posed problem.
  3. Yet we can still find a space $\mathcal{H}$ where $\mathcal{M} : \ell^2 \to \mathcal{H}$ is an isometry (i.e. with bounded inverse in particular). This space can be easily found to be

$$\mathcal{H} = \left\{ (a_n) \in \ell^2, \qquad \sum_{n\ge 0} |a_n|^2 e^{2n^2 T} < \infty \right\}.$$

[3] A linear operator between Hilbert spaces $A\colon \mathcal{H}_1 \to \mathcal{H}_2$ is *compact* if it maps bounded sequences to sequences with convergent subsequences.

[4] See the spectral theorem for bounded, self-adjoint operators

It corresponds to Fourier series which decay at an exponential rate. For such series, the corresponding Fourier cosine series $\sum_{k=0}^\infty a_k \cos(kx)$ is smooth on $[0,\pi]$. Then $\mathcal{M}$ maps rough functions into smooth ones. Undoing that process would then require "differentiating infinitely many times", one of the interpretations of severe ill-posedness.

Exercise 1. *For $k \ge 0$, define the scale of Hilbert spaces*

$$h^k := \left\{ \mathbf{u} = \{u_n\}_{n\ge 0} \in \mathbb{C}^{\mathbb{N}_0}, \ \|\mathbf{u}\|_k^2 := \sum_{n\ge 0} n^{2k}|u_n|^2 < \infty \right\}. \tag{3}$$

and denote $\ell^2 := h^0$. Consider a sequence of complex numbers $\{\lambda_n\}_{n\ge 0}$, and consider the 'diagonal' operator

$$A\colon \ell^2 \to \ell^2, \qquad A(\{u_n\}_n) = \{\lambda_n u_n\}_n.$$

1. *Under what condition is the operator A bounded (=continuous) ?*
2. *Suppose $\lambda_n \ne 0$ for all n. Is A invertible ? Is $A^{-1}\colon \ell^2 \to \ell^2$ always continuous ?*
3. *Fix $p \ge 0$ and suppose $\lambda_n = \frac{1}{n^p}$. With A defined as above, what is the order of ill-posedness of A in the scale h^k defined in (3) ?*
4. *Same question with the sequence $\lambda_n = e^{-n}$.*
5. *Can you describe the intersection space $h^\infty := \cap_{k\ge 0} h^k$?*

Exercise 2. *Consider the following measurement operator: given $f \in L^2([0,\pi])$, define $\mathcal{M}f = u|_{t=T}$ where $u(x,t)$ solves the wave equation*

$$\begin{aligned} \partial_{tt}u - \partial_{xx}u = 0, &\qquad x\in(0,\pi),\ t\in(0,T), \\ u(x,0) = f, &\qquad \partial_t u(x,0) = 0, \qquad 0\le x\le\pi, \\ \partial_x u(0,t) = \partial_x u(\pi,t) = 0, &\qquad 0 \le t \le T. \end{aligned}$$

1. *Using Fourier cosine series in x for f and u ($f \leftrightarrow \sum_{n=0}^\infty a_n\cos(nx)$), write an explicit expression for $\mathcal{M}$. Is $\mathcal{M} : L^2([0,\pi]) \to L^2([0,\pi])$ bounded ? injective ? surjective ? If $\widetilde{\mathcal{M}}$ describes $\mathcal{M}$ acting on sequences of Fourier cosine coefficients, and equip the space of such sequences with the Sobolev scale*

$$h^s := \left\{ (a_n)_{n\in\mathbb{N}}, \qquad \sum_{k=0}^\infty (k+1)^{2s}|a_k|^2 < \infty \right\},$$

does there exists $p \in \mathbb{R}_+$ such that $\widetilde{\mathcal{M}} : \ell^2 \to h^p$ is bounded and $\widetilde{\mathcal{M}}^{-1} : h^p \to \ell^2$ is bounded (two-sided estimate) ?

2. *Same as (a) replacing the initial conditions by*

$$u(x,0) = 0, \qquad \partial_t u(x,0) = f, \qquad 0 \le x \le \pi.$$

3. *Same as (a) replacing the wave equation with the elliptic equation $\partial_{tt} u + \partial_{xx} u = 0$.*

Exercise 3 (On moduli of stability)**.** *Let a* linear *forward operator* $\mathcal{M}\colon \mathcal{X} \to \mathcal{Y}$ *and* $\mathcal{X} = \cap_{k\ge 0}\mathcal{X}_k$, $\mathcal{Y} = \cap_{k\ge 0}\mathcal{Y}_k$ *with* $\{\mathcal{X}_k\}_k$, $\{\mathcal{Y}_k\}_k$ *Banach scales. Suppose that for every* k, $\mathcal{M}\colon \mathcal{X}_k \to \mathcal{Y}_k$ *is bounded. Assume moreover that* $\{\mathcal{Y}_k\}_k$ *satisfies* interpolation inequalities[5] *of the form, if* $0 \le k \le \ell$

$$\|g\|_{\mathcal{Y}_{\lambda k + (1-\lambda)\ell}} \le C \|g\|_{\mathcal{Y}_k}^{\lambda} \|g\|_{\mathcal{Y}_\ell}^{1-\lambda}, \qquad \lambda \in [0,1], \quad g \in \mathcal{Y}_\ell, \tag{4}$$

for some constant $C(k,\ell,\lambda)$*. Suppose that the operator* $\mathcal{M}$ *is ill-posed of order* $\alpha > 0$ *in the sense that*

$$\|x - x'\|_{\mathcal{X}_k} \le C \|\mathcal{M}(x) - \mathcal{M}(x')\|_{\mathcal{Y}_{k+\alpha}}, \quad k \ge 0. \tag{5}$$

Fixing k*, the question is whether we can obtain an estimate of* $\|x - x'\|_{\mathcal{X}_k}$ *in terms of a LOWER-regularity norm of* $\mathcal{M}(x - x')$ *than* $\mathcal{Y}_{k+\alpha}$*, ideally* $\mathcal{Y}_k$ *(as this might be the space where the noise lives). We show that this is possible if we add the* prior *assumption that the unknown has "high regularity", namely* $x, x' \in \mathcal{X}_\beta$ *for some* $\beta > \alpha + k$.

1. *Assuming the uniform bound* $\|x\|_{\mathcal{X}_\beta}, \|x'\|_{\mathcal{X}_\beta} \le C$ *for some* $\beta > \alpha + k$*, use* (4) *for appropriate* (λ, k, ℓ)*,* (5) *and the boundedness of* $\mathcal{M}$ *to show a Hölder estimate of the form*

$$\|x - x'\|_{\mathcal{X}_k} \le C \|\mathcal{M}(x) - \mathcal{M}(x')\|_{\mathcal{Y}_k}^{\theta},$$

   where $\theta \in [0,1]$ *depends on* α *and* β*.*

2. *How does* θ *behave as* β *increases ?*

3. *Use Hölder's inequality to show that the scale* $\{h^k\}_k$ *in* (3) *satisfies* (4)*.*

Exercise 4 (On Fourier series)**.** *Given* $f \in L^1(\mathbb{S}^1)$*, we may define the sequence of Fourier coefficients of* f*,* $\{c_n[f] := \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(\theta)\, d\theta\}_{n\in\mathbb{Z}}$*, a bounded, doubly infinite sequence (in* $\ell^\infty(\mathbb{Z})$*). For fixed* n*, we denote*

$$S_n[f](\theta) := \sum_{k=-n}^{n} c_k[f] e^{ik\theta} \in P_n,$$

where we denote P_n *the set of trigonometric polynomials of degree at most* n*, of the form* $p(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ *for some complex numbers* c_k*.*

On $[0, 2\pi]$*, let us define the Hilbert scale*

$$H^k(\mathbb{S}^1) = \left\{ f \in L^2(\mathbb{S}^1),\ \|f\|_k^2 := \sum_{j=0}^{k} \int_0^{2\pi} |f^{(j)}(\theta)|^2 < \infty \right\},$$

with $H^0 = L^2$*, equipped with the Hermitian inner product* $(f,g) = \int_0^{2\pi} f(\theta)\bar{g}(\theta)\, d\theta$*. On* $\mathbb{Z}$*, define the Hilbert scale*

$$h^k(\mathbb{Z}) = \left\{ \mathbf{u} \in \mathbb{C}^{\mathbb{Z}},\ \|\mathbf{u}\|_k^2 = \sum_{j\in\mathbb{Z}} (1 + j^2)^k |u_j|^2 < \infty \right\}.$$

[5] Sobolev scales most often satisfy this.

1. *Check that the functions* $\{\mathbf{e}_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}\}_{n\in\mathbb{Z}}$ *is an orthonormal system in* $L^2(\mathbb{S}^1)$ *and that* $S_n[f] = \sum_{k=-n}^n (f, \mathbf{e}_k)\mathbf{e}_k$. *In the sequel, we will assume that* $\{\mathbf{e}_n\}_{n\in\mathbb{Z}}$ *is a* complete[6] *orthonormal set in* $L^2(\mathbb{S}^1)$.

2. *Show that* $S_n[f]$ *is the minimizer of functional*

$$F(p) = \int_0^{2\pi} |f(\theta) - p(\theta)|^2 \, d\theta, \qquad p \in P_n.$$

   Thus, $S_n[f]$ *is the best* L^2*-approximant of* f *among all trigonometric polynomials of degree* n.

3. *Show that for every* n, $S_n[f]$ *and* $f - S_n[f]$ *are* L^2*-orthogonal and that*

$$\begin{aligned} \|f\|_{L^2}^2 &= \|S_n[f]\|_{L^2}^2 + \|f - S_n[f]\|_{L^2}^2 \\ &= 2\pi \sum_{k=-n}^{n} |c_k[f]|^2 + \|f - S_n[f]\|_{L^2}^2, \qquad n \ge 0. \end{aligned}$$

4. *Deduce that* $S_n[f]$ *converges to* f *in* $L^2(\mathbb{S}^1)$ *and that*

$$\|f\|_{L^2}^2 = 2\pi \sum_{k\in\mathbb{Z}} |c_k[f]|^2 \qquad (\textit{Parseval}). \tag{6}$$

   What does (6) *say about the Fourier series map* $L^2(\mathbb{S}^1) \ni f \mapsto \{c_k[f]\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ *?*

5. *Find the Fourier series of* $4\cos\theta\sin\theta$ *and* $\frac{1}{2+\cos^2\theta}$ *without computing integrals.*

6. *Suppose* $f \in C^1$. *Show that* $c_n[\frac{1}{i}\frac{d}{d\theta} f] = n\ c_n[f]$ *for all* $n \in \mathbb{Z}$.

7. *Given a polynomial* $q = \sum_{j=0}^{\ell} q_j x^j$, *denote* $q(\frac{1}{i}\frac{d}{d\theta}) := \sum_{j=0}^{\ell} q_j \left(\frac{1}{i}\frac{d}{d\theta}\right)^j$.

   Provided that $f \in C^\ell$, *show that* $c_n[q(\frac{1}{i}\frac{d}{d\theta})f] = q(n)c_n[f]$. *Combine this with* (6) *to deduce that* $f \in H^\ell(\mathbb{S}^1)$, *then its coefficients belong to* $h^\ell(\mathbb{Z})$.

8. *Prove the converse: if a sequence belongs to* $h^\ell(\mathbb{Z})$, *then the Fourier series construct a function that is* $H^\ell(\mathbb{S}^1)$.

9. *Cute stuff: Show that* $\frac{\pi^2}{6} = \sum_{k=1}^\infty \frac{1}{k^2}$ *by applying* (6) *to the function* $f(\theta) = \theta$.

10. *Find the integral kernel of the mapping* $f \mapsto S_n[f]$. *I.e., write* $S_n[f](\theta) = \int_0^{2\pi} K_n(\theta,\theta') f(\theta')\, d\theta'$ *for some function* $K_n(\theta,\theta')$ *(to be written in the simplest form possible).*

Exercise 5. *Let* $\{\lambda_n\}_n$ *a sequence of non-negative numbers decreasing to zero, and define the operator* $A\colon \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ *by* $(A\mathbf{u})_n = \lambda_n u_n$, $\mathbf{u} = \{u_n\}_n$.

Prove that the operator A *is compact.*

[6] This means: if $f \in L^2(\mathbb{S}^1)$ satisfies $(f, \mathbf{e}_n) = 0$ for all n, then $f = 0$. A proof can be found in [HN01, Ch. 7], showing that trigonometric polynomials are dense in $C(\mathbb{S}^1)$ using approximations of identity and the density of $C(\mathbb{S}^1)$ in $L^2(\mathbb{S}^1)$.

2 Explicit integral geometric examples

For the 'friendliest' cases of X-ray transforms on surfaces, one usually says that they are 'smoothing of order 1/2' and their associated normal operators are 'smoothing of order 1'. This section aims at making these statements most explicit, following one or more of the following routes:

- by computing an explicit singular value decomposition of the operator. Once this is done, the construction of Hilbert scales which reflect these smoothing properties becomes relatively straightforward.
- by finding an explicit functional relation between the X-ray transform (or rather, one of its normal operators) with distinguished differential operators. Since the latter oftentimes determine a scales of smoothness in their ambient spaces, the smoothing properties of the operator of interest naturally follows.

The next three examples include a compact manifold with boundary, a complete manifold, and a manifold with boundary. Each of these cases brings its own set of peculiarities.

2.1 The Funk transform on the two-sphere $\mathbb{S}^2$

2.1.1 Formulation

Our first example is the Funk transform, initially studied in [Fun16]. In the 70's, Guillemin [Gui76] used this transform in order to construct Zoll[7] metrics on the sphere. This example is also treated in [MP11, Sec. 1.2].

For the Funk transform, we can derive the full SVD of the operator, and appropriate Hilbert scales where to describe the mapping properties of the operator.

Let $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3, x^2+y^2+z^2 = 1\}$ the Euclidean 2-sphere. Given $f \in C^\infty(\mathbb{S}^2)$ and $p \in \mathbb{S}^2$, we define

$$If(p) := \int_0^{2\pi} f(\gamma_p(t))\ dt,$$

where γ_p is the equator on the sphere thinking of p as the North pole (it moves around !), traversed counterclockwise when viewed from p. If $p = N = (0,0,1)$ (the actual North Pole), $\gamma_N(t) = (\cos t, \sin t, 0)$. The parameterization of γ_p matters, and this one is chosen so that γ_p is a unit-speed geodesic on $\mathbb{S}^2$.

Exercise 6. *How to parameterize γ_p for any $p \in \mathbb{S}^2$?*

Since γ_p depends smoothly on p and $f \in C^\infty(\mathbb{S}^2)$, then $If \in C^\infty(\mathbb{S}^2)$. Note that in this case, f and If can both be viewed as functions on the same domain $\mathbb{S}^2$ (this will never happen again in the integral geometric problems considered below). The question to investigate is thus:

study the problem of reconstructing $f \in C^\infty(\mathbb{S}^2)$ from $If \in C^\infty(\mathbb{S}^2)$.

[7] A Zoll manifold is a closed Riemannian manifold, all of whose geodesics are closed and of the same length. The round sphere is such an example.

Obvious kernel and cokernel. Let $A : \mathbb{S}^2 \to \mathbb{S}^2$ the antipodal map ($A(p) = -p$). A is a smooth involution and induces a direct sum decomposition

$$C^\infty(\mathbb{S}^2) = C^\infty_{even}(\mathbb{S}^2) \oplus C^\infty_{odd}(\mathbb{S}^2),$$

where a function f will be considered even if $f \circ A = f$, odd if $f \circ A = -f$. Then the following two observations can be made: first $If = 0$ whenever f is odd, and If is even for any f. It therefore makes sense that we refine our initial question to:

$$\text{study the problem of reconstructing } f \in C^\infty_{even}(\mathbb{S}^2) \text{ from } If \in C^\infty_{even}(\mathbb{S}^2).$$

In what follows, we will show that I is an automorphism of $C^\infty_{even}(\mathbb{S}^2)$ and that, upon defining an appropriate Hilbert scale $H^s_{even}(\mathbb{S}^2)$, I is an isomorphism of H^s_{even} onto $H^{s+\frac{1}{2}}_{even}$. This will be based on understanding its eigendecomposition by means of spaces of spherical harmonics.

2.1.2 Spherical Harmonics

In order to define a scale of spaces on $\mathbb{S}^2$, one natural idea is to define, for $k = 2\ell$ even, $H^{2\ell}(\mathbb{S}^2)$ as the closure of $C^\infty(\mathbb{S}^2)$ for the topology defined by the norm $\|f\|_{H^{2\ell}} = \|(-\Delta_{\mathbb{S}^2} + 1)^\ell f\|_{L^2}$. Here $\Delta_{\mathbb{S}^2}$ denotes the Laplace-Beltrami operator[8], then $-\Delta_{\mathbb{S}^2}$ is non-negative, essentially self-adjoint operator and the $+1$ makes it injective. It is an example of an *elliptic* operator, which means that, in spite of the fact that this is a single differential operator, if Δf is L^2, then any second-order derivative of f is in L^2, although most other second-derivative ares are arguably "less natural" to write down.

When this is done, we realize that we only have spaces of even order, though we would like to define H^s for all integer s, or even for all $s \geq 0$ (and even all $s \in \mathbb{R}$). A process called *interpolation* allows to do this, though another clear picture emerges if we know the spectral decomposition of $\Delta_{\mathbb{S}^2}$. Note that general functional-analytic arguments show that $\Delta_{\mathbb{S}^2}$ has a complete, discrete eigensystem in $L^2(\mathbb{S}^2)$ and that its spectrum tends to ∞[9]. We now undertake this task, largely following [Fol95, Ch. 2.H].

The story of spherical harmonics sums up to this[10]: define $\mathcal{H}_k$ the space of polynomials on $\mathbb{R}^3$, harmonic[11] and homogeneous of degree k, and let $H_k = \{P|_{\mathbb{S}^2} : P \in \mathcal{H}_k\}$. The latter is called *spherical harmonics of degree k*.

Theorem 1. *(Spherical harmonics) (1) For every $k \in \mathbb{N}_0$, $H_k = \ker(-\Delta_{\mathbb{S}^2} - k(k+1)Id)$ and $\dim H_k = 2k+1$. Moreover, H_k is an irreducible $SO(3)$-module.*

---

[8] In the coordinates $(\theta, \varphi) \mapsto (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, $\Delta_{\mathbb{S}^2} f = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial f}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$

[9] Sketch of proof: by Riesz-representation theorem, for $f \in L^2$, the problem $-\Delta_{\mathbb{S}^2} u + u = -f$ admits a unique solution in H^1, this defines $(-\Delta_{\mathbb{S}^2} + 1)^{-1} \colon L^2(\mathbb{S}^2) \to H^1(\mathbb{S}^2)$ as a bounded operator. Since the inclusion $H^1 \to L^2$ is compact, then $(-\Delta_{\mathbb{S}^2} + 1)^{-1}$ is a compact, operator, moreover, injective and self-adjoint. By the spectral theorem for compact, self-adjoint operators, there exists a complete orthonormal set ϕ_n of $L^2(\mathbb{S}^2)$ along with a decreasing sequence $\lambda_n \to 0$ such that $(-\Delta_{\mathbb{S}^2} + 1)^{-1}\phi_n = \lambda_n\phi_n$. Then $(\phi_n, \lambda_n^{-1} - 1)$ is an eigensystem for $\Delta_{\mathbb{S}^2}$.

[10] it works with minor modifications in all dimensions ≥ 2, see [Fol95, Ch. 2.H]

[11] in the sense that $\Delta_{\mathbb{R}^3} P = 0$, where $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$

(2) We have the direct orthogonal sum

$$L^2(\mathbb{S}^2) = \bigoplus_{k\geq 0} H_k, \tag{7}$$

in the sense that every $f \in L^2(\mathbb{S})$ admits a unique orthogonal and L^2-convergent decomposition $f = \sum_{k\geq 0} f_k$ with $f_k \in H_k$.

Proof of Theorem 1. First let $\mathcal{P}_k$ the space of polynomials on $\mathbb{R}^3$, homogeneous of degree k, and write $r^2 := x^2+y^2+z^2$. One may show as exercise that $\dim \mathcal{P}_k = (k+1)(k+2)/2$. The first question is to understand the orthocomplement of $\mathcal{H}_k$ in $\mathcal{P}_k$. To this effect, [Fol95, Prop. 2.49] states that for $k \geq 2$,

$$\mathcal{P}_k = \mathcal{H}_k \oplus r^2\mathcal{P}_{k-2}, \qquad r^2\mathcal{P}_{k-2} := \{r^2 P : P \in \mathcal{P}_{k-2}\}, \tag{8}$$

whose proof is given in Ex. 7. As a result,

$$\dim \mathcal{H}_k = \dim \mathcal{P}_k - \dim \mathcal{P}_{k-2} = 2k+1,$$

and by induction,

$$\mathcal{P}_k = \mathcal{H}_k \oplus r^2\mathcal{H}_{k-2} \oplus r^4\mathcal{H}_{k-4} \dots \tag{9}$$

Also recall the expression of the Laplacian in spherical coordinates:

$$\Delta_{\mathbb{R}^3} = \frac{1}{r^2}\Delta_{\mathbb{S}^2} + \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}.$$

Now, if $f \in \mathcal{H}_k$, then one may write $f = r^k \bar{f}$ where $\bar{f} = f|_{\mathbb{S}^2}$, then applying the equation above and evaluating at $r=1$ yields the relation

$$\Delta_{\mathbb{R}^3} f = 0 = \Delta_{\mathbb{S}^2}\bar{f} + k(k+1)\bar{f}.$$

Hence H_k consists is the eigenspace of $\Delta_{\mathbb{S}^2}$ associated with eigenvalue $-k(k+1)$. As $\Delta_{\mathbb{S}^2}$ is self-adjoint, this implies the $L^2(\mathbb{S}^2)$-orthogonality $H_k \perp H_\ell$ for $k \neq \ell$.

The proof of (7) is based on the Weierstrass approximation theorem, together with (9): in a nutshell, a function in $L^2(\mathbb{S}^2)$ can be approximated by functions in $C(\mathbb{S}^2)$, which in turn can be approximated by restrictions to $\mathbb{S}^2$ of polynomials, which by (9) decompose as finite sums of spherical harmonics. □

Exercise 7. *This problem guides you through the proof of (8). Here and below, denote $\mathrm{x} = (x,y,z)$ and for a tri-index $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{N}_0^3$, we denote $|\alpha| = \alpha_1+\alpha_2+\alpha_3$, $\alpha! = \alpha_1!\alpha_2!\alpha_3!$, $\mathrm{x}^\alpha = x^{\alpha_1}y^{\alpha_2}z^{\alpha_3}$ and $\partial^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_z^{\alpha_3}$. Thus, a general element of $\mathcal{P}_k$ takes the form $P = \sum_{|\alpha|=k} a_\alpha \mathrm{x}^\alpha$ for some complex numbers $\{a_\alpha\}_\alpha$, where the sum runs over all tri-indices of length k, and for such a P, we define the differential operator $P(\partial) := \sum_{|\alpha|=k} a_\alpha \partial^\alpha$.*

1. *On $\mathcal{P}_k$, we define the inner product*

$$\left(P = \sum_{|\alpha|=k} a_\alpha \mathrm{x}^\alpha, Q = \sum_{|\beta|=k} b_\beta \mathrm{x}^\beta\right) \mapsto \sum_{|\alpha|=k} \alpha! a_\alpha \overline{b_\alpha}.$$

   Show that such an inner product can be obtained by computing the quantity $\{P, Q\} := P(\partial)\overline{Q}$. *[Hint: show that* $\{\mathrm{x}^\alpha, \mathrm{x}^\beta\} = \alpha!$ *if* $\alpha = \beta$, 0 *otherwise.]*

2. *Show that for* $P \in \mathcal{P}_{k-2}$ *and* $Q \in \mathcal{P}_k$,

$$\{r^2 P, Q\} = \{P, \Delta_{\mathbb{R}^3} Q\}.$$

3. *Conclude* (8).

2.1.3 Eigendecomposition of the Funk transform

The group $SO(3)$ acts on $C^\infty(\mathbb{S}^2)$ by rotations: $g\cdot f(p) = f(g^{-1}p)$. The key observation is that I commutes[12] with this action in the sense that

$$I(g\cdot f) = g\cdot If, \qquad g\in SO(3), \qquad f\in C^\infty(\mathbb{S}^2).$$

By *Schur's lemma*, this implies two things at the level of all the irreducible $SO(3)$-modules H_k: for all $k\ge 0$,

- $I(H_k)\subseteq H_k$
- $I|_{H_k} = \lambda_k Id$ for some constant λ_k.

If k is odd, since $H_k\subset C^\infty_{odd}(\mathbb{S}^2)$, we already know that $\lambda_k = 0$. For k even, $k=2\ell$ for some $\ell\ge 0$, it suffices to find a "good" function in $f\in H_{2\ell}$ and a point p where $f(p)\ne 0$, then $\lambda_{2\ell}$ will be given by $c_{2\ell} = \frac{If(p)}{f(p)}$. A good choice is as follows: one may check that the "sectoral" harmonic function $f(x,y,z) = (x+iy)^{2\ell}$, restricted to $\mathbb{S}^2$, belongs to $H_{2\ell}$. Now choose $p=(1,0,0)$, with $\gamma_p(t) = (0,\cos t,\sin t)$ to deduce

$$\lambda_{2\ell} = \frac{1}{f(1,0,0)}\int_0^{2\pi} f(0,\cos t,\sin t)\ dt = (-1)^\ell \int_0^{2\pi} (\cos t)^{2\ell}\ dt.$$

The above is a *Wallis integral*, and one may deduce the final expression

$$\lambda_{2\ell} = (-1)^\ell 2\pi \frac{(2\ell)!}{2^{2\ell}(\ell!)^2}, \qquad \ell\in\mathbb{N}_0. \tag{10}$$

As a conclusion, the eigenvalue decomposition of $I\colon L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)$ is given by:

$$\ker I = \bigoplus_{\ell\ge 0} H_{2\ell+1}, \quad \ker(I-\lambda_{2\ell}) = H_{2\ell}, \qquad \ell\ge 0,$$

or at the level of the spectral decomposition:

$$If = \sum_{k\ge 0}\lambda_k f_k, \qquad f = \sum_{k\ge 0} f_k, \qquad f_k\in H_k.$$

Using *Stirling's formula*[13], we arrive at the conclusion that

$$|\lambda_{2\ell}| \sim \sqrt{8\pi}(2\ell)^{-1/2}, \qquad \ell\to\infty. \tag{11}$$

We now explain how to exploit this to formulate mapping properties of the Funk transform in a sharp way.

[12] In representation theory language, I is equivariant w.r.t the $SO(3)$ action, or I is $SO(3)$-linear.
[13] $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ as $n\to\infty$

2.1.4 Mapping properties

To formulate mapping properties, we need two things: (1) to translate smoothness on $\mathbb{S}^2$ into rate of decay of the spherical harmonic decomposition and (2) to use the asymptotics of $\lambda_{2\ell}$ for large ℓ given in (11).

For $s \geq 0$, one may define $H^s(\mathbb{S}^2)$ to be the closure of $C^\infty(\mathbb{S}^2)$ for the norm

$$\|f\|_{H^s}^2 := \sum_{k\geq 0} (1+k(k+1))^s \|f_k\|_{L^2}^2, \qquad f = \sum_{k\geq 0} f_k, \quad f_k \in H_k. \tag{12}$$

For $s = 2\ell$ an even integer, this amounts to the norm $\|(-\Delta+1)^\ell f\|_{L^2}$ and hence it indeed encodes that, when this quantity is finite, the function f has 2ℓ square-integrable derivatives. Allowing s to be any non-negative reals achieves a few things: it gives meaning to "having derivatives of fractional order"; it hints at the fact that for any $s \geq 0$, the operator

$$f = \sum_{k\geq 0} f_k \mapsto \sum_{k\geq 0} (1+k(k+1))^{s/2} f_k$$

is one way to define the operator $(-\Delta+1)^{s/2}$. By construction, this operator is $H^s \to L^2$ bounded, a process which embodies the action of taking "s derivatives" (in an "isotropic" way).

Remark 1. *Given s fixed, although the construction (12) has the special property that it can be related exactly to the operator $(-\Delta+1)$, one may notice that for* any *sequence d_k such that (i) $d_k \neq 0$ for all k and (ii) there exists two positive constants such that $C_1 \leq |k^s d_k| \leq C_2$, one may define the norm*

$$\|f\|^2 := \sum_{k\geq 0} d_k^2 \|f_k\|_{L^2}^2, \qquad f = \sum_{k\geq 0} f_k, \quad f_k \in H_k,$$

and the closure of $C^\infty(\mathbb{S}^2)$ with respect to that norm would give a Hilbert space whose topology is the same as $H^s(\mathbb{S}^2)$.

Just like $C^\infty = C^\infty_{even} \oplus C^\infty_{odd}$, we can split these Sobolev spaces into even and odd functions

$$H^s(\mathbb{S}^2) = H^s_{even}(\mathbb{S}^2) \oplus H^s_{odd}(\mathbb{S}^2).$$

On to the smoothing properties of I, the main crux is to understand the polynomial behavior of $\lambda_{2\ell}$ as $\ell \to \infty$. By (11), there exists two positive constants C_1, C_2 such that

$$C_1(1+(2\ell)^{1/2}) \leq |\lambda_{2\ell}| \leq C_2(1+(2\ell)^{1/2}), \qquad \ell \geq 0. \tag{13}$$

Out of this, we can deduce the two-sided estimates

$$C_1 \|f\|_{H^s_{even}} \leq \|If\|_{H^{s+\frac{1}{2}}_{even}} \leq C_2 \|f\|_{H^s_{even}}, \tag{14}$$

a precise description of the fact that the operator I is smoothing of order $\frac{1}{2}$.

Exercise 8. *Work out the details of (13) and (14).*

2.2 The Radon transform on the plane $\mathbb{R}^2$

[Recalls on Schwartz space and tempered distributions - see PDE lecture notes]

Although we will spend more time with the Radon transform in the lectures that follow, the main point of this section is to show that this example can be naturally tackled using Fourier analysis on $\mathbb{R}^2$. Given $f \in C_c(\mathbb{R}^2)$ or $f \in \mathscr{S}(\mathbb{R}^2)$, we define

$$Rf(s,\theta) = \int_{\mathbb{R}} f(s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp)\ dt, \qquad (s,\theta) \in \mathcal{Z} := \mathbb{R}\times\mathbb{S}^1, \tag{15}$$

where $\boldsymbol{\theta} = \binom{\cos\theta}{\sin\theta}$ and $\boldsymbol{\theta}^\perp = \binom{-\sin\theta}{\cos\theta}$. One may equip $\mathcal{Z}$ with area element $ds\ d\theta$ and define $L^2(\mathcal{Z})$ with respect to that measure.

2.2.1 Basic mapping observations

One can define a Schwartz space on $\mathscr{S}(\mathcal{Z})$ and show that $R\colon \mathscr{S}(\mathbb{R}^2) \to \mathscr{S}(\mathcal{Z})$ is continuous.

Similarly, $\mathscr{S}\colon C_c^\infty(\mathbb{R}^2) \to C_c^\infty(\mathcal{Z})$ is continuous[14]. Observe in particular that if f is supported in $\{|x| < R\}$, then Rf is supported in the truncated cylinder $\{|s| < R\}$.

2.2.2 The $L^2 - L^2$ adjoint R^*

By direct calculation, one may compute the $L^2(\mathbb{R}^2) \to L^2(\mathcal{Z})$ adjoint R^* given by

$$R^*g(\mathrm{x}) = \int_{\mathbb{S}^1} g(\mathrm{x}\cdot\boldsymbol{\theta}, \theta)\ d\theta, \qquad \mathrm{x}\in\mathbb{R}^2.$$

Notice the duality of geometries: the Radon transform integrates a function over all points through a line, while its transpose integrates over all lines through a point.

2.2.3 The normal operator R^*R as a convolution operator

Combining both definitions of R and R^*, we compute directly that

$$R^*Rf(\mathrm{x}) = 2\int_{\mathbb{R}^2} f(\mathrm{y})\frac{1}{|\mathrm{x}-\mathrm{y}|}\ d\mathrm{y} = \left(\frac{2}{|\cdot|}\star f\right)(\mathrm{x}).$$

Since R^*R is a convolution operator, is becomes diagonalized through the Fourier transform, more specifically

$$\widehat{R^*Rf}(\xi) = 2\hat{h}(\xi)\hat{f}(\xi), \qquad h(\mathrm{x}) := \frac{1}{|\mathrm{x}|}.$$

The question is: what sense does the Fourier transform of h make, and how to compute it ?

[14] Note that spaces of smooth functions with compact support are slightly more general than Fréchet.

Indeed, h is not in L^1, nor in L^2... On the other hand, it makes sense as a tempered distribution, since for $f \in \mathscr{S}(\mathbb{R}^2)$,

$$\left| \int_{\mathbb{R}^2} \frac{1}{|\mathrm{x}|} f(\mathrm{x})\ d\mathrm{x} \right| \le \sup_{\mathrm{x}} |f(\mathrm{x})(1+|\mathrm{x}|^2)| \int_{\mathbb{R}^2} \frac{1}{|\mathrm{x}|(1+|\mathrm{x}|^2)}\ d\mathrm{x} \le C \sup_{\mathrm{x}} |f(\mathrm{x})(1+|\mathrm{x}|^2)|.$$

This justifies that its Fourier transform makes sense as a tempered distribution. To compute it, we use the following trick: the function h is the $\mathscr{S}'$-limit of $h_\epsilon(\mathrm{x}) = \frac{e^{-\varepsilon|x|}}{|\mathrm{x}|}$ in the sense that for every $f \in \mathscr{S}$, $\langle h_\epsilon, f\rangle_{\mathscr{S}',\mathscr{S}} \to \langle h, f\rangle_{\mathscr{S}',\mathscr{S}}$ as $\epsilon \to 0$. Since the Fourier transform is $\mathscr{S}' \to \mathscr{S}'$-continuous, $\hat{h}$ is the $\mathscr{S}'$-limit of $\widehat{h_\epsilon}$, and since $h_\epsilon \in L^1(\mathbb{R}^2)$, we can compute its Fourier transform in the 'classical' sense:

$$\begin{aligned} \widehat{h_\epsilon}(\xi) = \int_{\mathbb{R}^2} \frac{e^{-\epsilon|\mathrm{x}|}}{|\mathrm{x}|} e^{-i\mathrm{x}\cdot\xi}\ d\mathrm{x} = \int_0^\infty \int_{\mathbb{S}^1} e^{-\epsilon\rho - i\rho|\xi|\cos\theta}\ d\theta\ d\rho &= \int_{\mathbb{S}^1} \frac{1}{\epsilon + i|\xi|\cos\theta}\ d\theta \\ &= \frac{1}{|\xi|} \int_{\mathbb{S}^1} \frac{1}{\frac{\epsilon}{|\xi|} + i\cos\theta}\ d\theta. \end{aligned}$$

Now by complex integration, one can show that for any $a > 0$,

$$\int_{\mathbb{S}^1} \frac{d\theta}{a + i\cos\theta} = \frac{2\pi}{\sqrt{1+a^2}}, \tag{16}$$

and hence,

$$\widehat{h_\epsilon}(\xi) = \frac{2\pi}{|\xi|} \frac{1}{\sqrt{1 + \frac{\epsilon^2}{|\xi|^2}}},$$

whose pointwise limit (and hence in $\mathscr{S}'$) is $\hat{h}(\xi) = \frac{2\pi}{|\xi|}$.

Conclusion. We then conclude that

$$\widehat{R^*Rf}(\xi) = \frac{4\pi}{|\xi|} \hat{f}(\xi).$$

In particular, note that since $\widehat{-\Delta f}(\xi) = |\xi|^2 \hat{f}(\xi)$, then

$$\mathcal{F}((R^*R)^2(-\Delta)f)(\xi) = \frac{(4\pi)^2}{|\xi|^2} |\xi|^2 \hat{f}(\xi) = (4\pi)^2 \hat{f}(\xi).$$

In other words, $(R^*R)^2(-\Delta)f = (4\pi)^2 f$, a statement which one may think of as "the operator $\frac{1}{4\pi} R^*R$ is a negative squareroot of $(-\Delta)$".

What remains to clarify is: in what spaces does all of this work ?

Exercise 9. *Prove* (16).

Exercise 10 (Basic properties of the Radon transform)**.** *In what follows, for $r \ge 0$, we denote $B_r = \{\mathrm{x} \in \mathbb{R}^2 : |\mathrm{x}| \le r\}$.*

1. *Show that if $f \in C_c^\infty(\mathbb{R}^2)$ vanishes outside B_r, then Rf is smooth with compact support, and vanishes outside $\{(s,\theta) \in \mathcal{Z},\ |s| \le r\}$.*
2. *Show that for $n \ge 2$, if $f(z) = z^{-n}$ $(z = x+iy)$, then $Rf \equiv 0$. In particular, the converse to the previous statement is false.*
3. *Show that $R\colon L^2(B_r) \to L^2([-r,r]\times\mathbb{S}^1)$ is bounded.*

Exercise 11. *Let $f \in \mathcal{S}(\mathbb{R}^2)$.*

1. *Show that $R[\partial_x f] = \cos\theta \frac{\partial}{\partial s} Rf$ and $R[\partial_y f] = \sin\theta \frac{\partial}{\partial s} Rf$.*
2. *Show that $\frac{\partial^2}{\partial s^2} Rf(s,\theta) = R[\Delta f](s,\theta)$.* $\qquad \Delta = \partial_x^2 + \partial_y^2$.

Exercise 12. 1. *Compute the Radon transform of the characteristic function of the ball of radius $\epsilon > 0$ and center $s_0 e^{i\beta}$ with $\rho > 0$.*

2. *Let $f(\mathrm{x}) = \begin{cases} (1-|\mathrm{x}|^2)^{-1/2} & |\mathrm{x}| < 1, \\ 0 & |\mathrm{x}| \ge 1. \end{cases}$ Compute $Rf(s,\theta)$ for $|s| \le 1$.*

Exercise 13. *For $\alpha \in \mathbb{R}$, define the weighted L^2 space*

$$L^{2,\alpha}(\mathbb{R}^2) = \left\{ f : \int |f(\mathrm{x})|^2 (1+|\mathrm{x}|^2)^\alpha \ d\mathrm{x} < \infty \right\}.$$

For what α's do we have that $R : L^{2,\alpha}(\mathbb{R}^2) \to L^2(\mathcal{Z})$ is bounded ?

Exercise 14. *If a function is supported inside the unit disk $\mathbb{D}$, we may then define its Radon transform on the "truncated cylinder" $[-1,1]\times\mathbb{S}^1$, equipped with the measure $dsd\theta$.*

1. *On the unit disk $\mathbb{D}$, define the weighted space*

$$L^{2,\alpha}(\mathbb{D}) = \left\{ f : \int |f(\mathrm{x})|^2 (1-|\mathrm{x}|^2)^\alpha \ d\mathrm{x} < \infty \right\}.$$

For what α do we have that $R\colon L^{2,\alpha}(\mathbb{D}) \to L^2([-1,1]\times\mathbb{S}^1)$ is bounded ?

2. *On the truncated cylinder, define the weighted space*

$$L^{2,\beta}([-1,1]\times\mathbb{S}) = \left\{ g : \int |g(s,\theta)|^2 (1-s^2)^\beta \ dsd\theta < \infty \right\}.$$

For which β do we have that $R\colon L^2(\mathbb{D}) \to L^{2,\beta}([-1,1]\times\mathbb{S}^1)$ is bounded ?

2.3 The X-ray transform on the unit disk $\mathbb{D}$

While we will leave for later the question of finding the right spaces for the Radon transform (mainly because the continuous spectrum aspect of the problem raises additional issues). We will now return to an example where the integral geometric operator has discrete spectrum, and where a Singular Value Decomposition can be derived.

2.3.1 From parallel beam to fan-beam

From Exercise (14), one may find out that the operator

$$R\colon L^2(\mathbb{D}) \to L^2([-1,1]\times\mathbb{S}^1, (1-s^2)^{-1/2}\ ds\ d\theta)$$

is bounded. In spirit this weight on the codomain arises from the fact that integration curves become shorter as $|s|$ approaches 1 and this also contributes to the smallness of Rf there. Note that we have shrunk the co-domain, and thus we have changed the expression of the adjoint, which now looks like

$$R^* g(\mathrm{x}) = \int_{\mathbb{S}^1} [(1-s^2)^{-1/2} g](\mathrm{x}\cdot\boldsymbol{\theta}, \theta)\ d\theta = \int_{\mathbb{S}^1} \frac{g(\mathrm{x}\cdot\boldsymbol{\theta},\theta)}{(1-(\mathrm{x}\cdot\boldsymbol{\theta})^2)^{1/2}}\ d\theta.$$

There are "natural" angle variables in which the measure $(1-s^2)^{-1/2}\ ds\ d\theta$ becomes $d\alpha\ d\beta$, called **fan-beam coordinates**: $\beta\in\mathbb{S}^1$ parameterizes a point from the boundary, while $\alpha\in(-\pi/2,\pi/2)$ parameterizes the angle where to cast the line segment, relative to the inner pointing normal. See Fig. 1 for an example of both transforms side-by-side.

We now call I_0 the operator $R|_{L^2(\mathbb{D})}$, for reasons that may appear more obvious later[15]. Hence, for $f\in L^2(\mathbb{D})$, let us define

$$I_0 f(\beta,\alpha) := \int_0^{2\cos\alpha} f(e^{i\beta} + te^{i(\beta+\pi+\alpha)})\ dt, \qquad (\beta,\alpha)\in\mathbb{S}^1\times[-\pi/2,\pi/2]. \tag{17}$$

Since $I_0(\beta,\alpha) = Rf(\sin\alpha, \beta+\pi/2+\alpha)$, using a change of variable from (s,θ) to (β,α), one may show that $I_0\colon L^2(\mathbb{D})\to L^2(\mathbb{S}^1_\beta\times[-\pi/2,\pi/2]_\alpha,\ d\alpha\ d\beta)$ is bounded, and its adjoint (a.k.a. *the backprojection operator*) takes the form

$$I_0^* g(\mathrm{x}) = \int_{\mathbb{S}^1} \frac{1}{\cos\alpha_-(\mathrm{x},\theta)} g(\beta_-(\mathrm{x},\theta), \alpha_-(\mathrm{x},\theta))\ d\theta, \tag{18}$$

where $\beta_-(\mathrm{x},\theta)$, $\alpha_-(\mathrm{x},\theta)$ are the fan-beam coordinates of the unique line passing through (x,θ), as explained in section 2.3.3 below.

The purpose below is to compute the full SVD of I_0 using the method of intertwining differential operators, which we now recall.

[15] If you can't wait: I is notation for the geodesic X-ray transform on functions on the unit tangent bundle SM, while I_0 is the restriction of the former to functions on M.

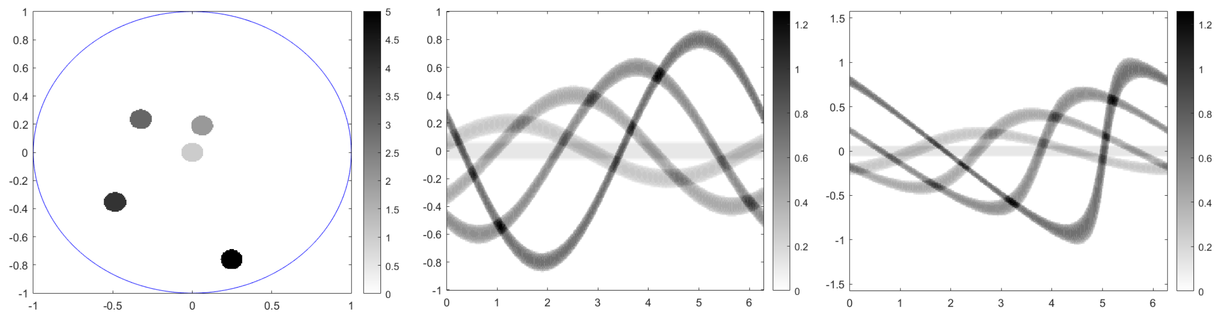

Figure 1: left to right: a function f, its Radon transform Rf (axes: (θ, s)), its X-ray transform $I_0 f$ (axes: (β, α))

2.3.2 The method of intertwining operators

Suppose we have a bounded, injective operator between two Hilbert spaces

$$A\colon (\mathcal{H}_1, \|\cdot\|_1) \to (\mathcal{H}_2, \|\cdot\|_2),$$

and suppose we have two self-adjoint operators $D_1\colon \mathcal{D}(D_1) \to \mathcal{H}_1$ and $D_2\colon \mathcal{D}(D_2) \to \mathcal{H}_2$, defined on dense subspaces $\mathcal{D}(D_1) \subset \mathcal{H}_1$ and $\mathcal{D}(D_2) \subset \mathcal{H}_2$[16] such that $A(\mathcal{D}(D_1)) \subset \mathcal{D}(D_2)$.

Suppose that the *intertwining relation* $A \circ D_1 = D_2 \circ A$ holds on $\mathcal{D}(D_1)$, and assume further that D_1 has simple spectrum $\lambda_0 \le \lambda_1 \le \dots \le \lambda_n \le \dots$ with eigenvectors $\{u_n\}_{n\ge 0}$, a complete orthonormal set in $\mathcal{H}_1$.

Theorem 2. *With the assumptions above, the SVD of $A\colon \mathcal{H}_1 \to A(\mathcal{H}_1)$ is given by*

$$\left(\frac{u_n}{\|u_n\|_1}, \frac{Au_n}{\|Au_n\|_2}, \frac{\|u_n\|_1}{\|Au_n\|_2} \right)_{n\ge 0}.$$

Note that A is not always surjective, this is why the operator is co-restricted to its range in the statement.

Proof of Theorem 2. For all $n \ge 0$, set $v_n = Au_n$. We have that

$$D_2 v_n = D_2(Au_n) = A(D_1 u_n) = A(\lambda_n u_n) = \lambda_n v_n,$$

hence the family $\{v_n\}_{n\ge 0}$, being eigenvectors with distinct eigenvalues of the self-adjoint operator, is an orthogonal family. By definition, $\{v_n/\|v_n\|_2\}_{n\ge 0}$ is a complete orthonormal set in $A(\mathcal{H}_1)$, and we obviously have

$$A\frac{u_n}{\|u_n\|_1} = \frac{\|v_n\|_2}{\|u_n\|_1} \frac{v_n}{\|v_n\|_2}, \qquad n \ge 0.$$

[16] This is usually the case of differential operators and when the space $\mathcal{H}$ is an L^2 space. Such operators are never bounded, but they are densely defined (say, on C^∞), and the theory of unbounded operators still allows for their good understanding.

It just remains to show that

$$A^* \frac{v_n}{\|v_n\|_2} = \frac{\|v_n\|_2}{\|u_n\|_1} \frac{u_n}{\|u_n\|_1}, \qquad n \ge 0.$$

To do this, we simply expand $A^* v_n = \sum_{p\ge 0} a_{np} u_p$, where

$$a_{np} = \frac{\langle A^* v_n, u_p\rangle}{\|u_p\|_1^2} = \frac{\langle v_n, A u_p\rangle}{\|u_p\|_1^2} = \frac{\langle v_n, v_p\rangle}{\|u_p\|_1^2} = \delta_{np} \frac{\|v_n\|_2^2}{\|u_n\|_1^2},$$

and the result follows. □

The original idea can be found in [Maa91], where the backprojection operator I_0^* is first diagonalized using spherical harmonics, into countably many one-dimensional operators (one for each spherical mode). Each such operator intertwines two second-order differential operators, and the computation of the spectrum of one involves solving ODEs, which in spirit is much simpler than solving integral equations.

What if A is not injective ? Note further that upon taking adjoints and using the self-adjointness of D_1, D_2, we may obtain a second intertwining relation $D_1 \circ A^* = A^* \circ D_2$, and combining the two, we arrive that the fact that $[A^*A, D_1] = 0$ and $[AA^*, D_2] = 0$.

In particular, the eigenspaces of D_1 are A^*A-stable, and since they are all one-dimensional $A^*Au_n = a_n u_n$ for every $n \ge 0$. The kernel of A is precisely the span of those vectors u_n for which $a_n = 0$, and upon removing those and replacing A by its injective restriction $A|_{\{\ker A\}^\perp}$, we can apply Theorem 2.

In what follows, we will **factor in** the circular symmetry of the problem by considering **pairs** of intertwined differential operators.

2.3.3 Interwiners for the backprojection operator I_0^*

The presentation that follows is a combination of [Mon20, Section 3] and [MM19]. We will denote

$$\partial_+ S\mathbb{D} = \mathbb{S}^1_\beta \times [-\pi/2, \pi/2]_\alpha, \qquad \mu = \cos\alpha.$$

Let us define the operator $I_0^\sharp \colon C_\alpha^\infty(\partial_+ S\mathbb{D}) \to C^\infty(\mathbb{D})$ as the formal adjoint of $I_0 : L^2(\mathbb{D}) \to L^2(\partial_+ S\mathbb{D}, \mu\ d\alpha\ d\beta)$, (so that I_0^* defined in (18) takes the form $I_0^* := I_0^\sharp(\frac{1}{\mu}\cdot))$. Such an operator takes the form

$$I_0^\sharp g(\mathrm{x}) = \int_{\mathbb{S}^1} g(\beta_-(\mathrm{x},\theta), \alpha_-(\mathrm{x},\theta))\ d\theta, \tag{19}$$

where $\beta_-(\mathrm{x},\theta)$, $\alpha_-(\mathrm{x},\theta)$ are the fan-beam coordinates of the unique line passing through (x,θ). In what follows, we will identify x with $\rho e^{i\omega}$. See Figure 2 for a summary.

From the observation made in Fig. 2, these functions satisfy the following relation:

$$\beta_-(\rho e^{i\omega}, \theta) = \omega + \beta_-(\rho, \theta - \omega), \qquad \alpha_-(\rho e^{i\omega}, \theta) = \alpha_-(\rho, \theta - \omega).$$

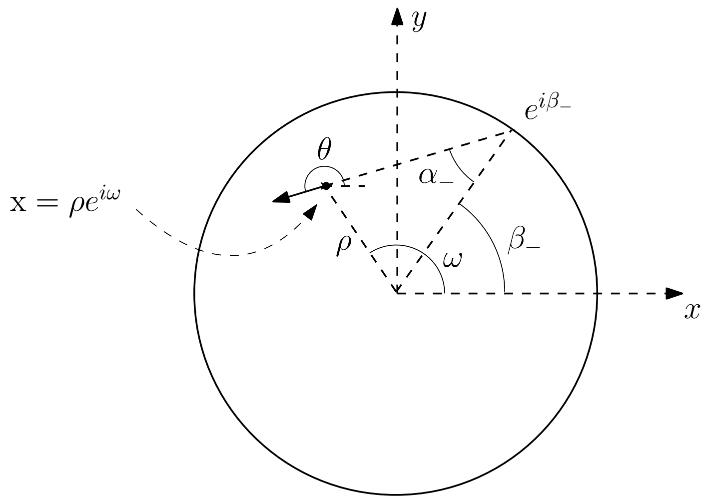

Figure 2: Setting of definition of $(\beta_-(\rho e^{i\omega}, \theta), \alpha_-(\rho e^{i\omega}, \theta))$ (written as (β_-, α_-) on the diagram). The rotation invariance implies that if θ and ω are translated by δ, then β_- is translated by δ and α_- remains unchanged.

In particular, the expression of $I_0^\sharp g$ immediately becomes

$$I_0^\sharp g(\rho e^{i\omega}) = \int_{\mathbb{S}^1} g(\omega + \beta_-(\rho, \theta - \omega), \alpha_-(\rho, \theta - \omega))\ d\theta = \int_{\mathbb{S}^1} g(\omega + \beta_-(\rho, \theta), \alpha_-(\rho, \theta))\ d\theta.$$

We then immediately see the first intertwining property

$$\partial_\omega \circ I_0^\sharp = I_0^\sharp \circ \partial_\beta, \qquad \partial_\omega \circ I_0^* = I_0^* \circ \partial_\beta.$$

Upon defining

$$T := \partial_\beta - \partial_\alpha, \tag{20}$$

a second intertwining property is then given as follows.

Theorem 3. *Define the operators*

$$L := (1-\rho^2)\frac{\partial^2}{\partial \rho^2} + \left(\frac{1}{\rho} - 3\rho\right)\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \omega^2}, \tag{21}$$

and $D := T^2 + 2\tan\alpha T$. Then we have the following intertwining properties:

$$L \circ I_0^\sharp = I_0^\sharp \circ D, \tag{22}$$

$$\mathcal{L} \circ I_0^* = I_0^* \circ (-T^2), \qquad \mathcal{L} := -L + 1. \tag{23}$$

Proof. Proof of (22). In what follows, α_- and β_- will be short for $\alpha_-(\rho, \theta)$ and $\beta_-(\rho, \theta)$. Note the easy two properties

$$\beta_- + \alpha_- + \pi = \theta, \qquad \sin\alpha_- = -\rho\sin\theta.$$

In particular, this gives $\frac{\partial \alpha_-}{\partial \rho} = -\frac{\sin\theta}{\cos\alpha_-} = \frac{1}{\rho}\tan\alpha_-$, $\frac{\partial\alpha_-}{\partial\theta} = \frac{-\rho\cos\theta}{\cos\alpha_-}$, and the derivatives of β_- can be deduced through the relations

$$\frac{\partial \beta_-}{\partial \rho} = -\frac{\partial \alpha_-}{\partial \rho}, \qquad \frac{\partial \beta_-}{\partial \theta} = 1 - \frac{\partial \alpha_-}{\partial \theta}.$$

From these relations, we immediately deduce the property that

$$\frac{\partial}{\partial \rho} I_0^\sharp g = -\frac{1}{\rho} I_0^\sharp[\tan\alpha T g].$$

Iterating this formula, we obtain

$$\frac{\partial^2}{\partial \rho^2} I_0^\sharp g = \frac{1}{\rho^2} I_0^\sharp[\tan\alpha T g] + \frac{1}{\rho^2} I_0^\sharp[\tan\alpha T(\tan\alpha T g)] = \frac{1}{\rho^2} I_0^\sharp[\tan^2\alpha T^2 g - \tan^3\alpha T g].$$

Then by direct algebra, using the last two identities, we obtain

$$\begin{aligned} [(1-\rho^2)\partial_\rho^2 + (\frac{1}{\rho} - 3\rho)\partial_\rho] I_0^\sharp g = \frac{1}{\rho^2} I_0^\sharp[\tan^2\alpha T^2 g - \tan\alpha(1+\tan^2\alpha) T g] \dots \\ - I_0^\sharp[\tan^2\alpha T^2 g - \tan\alpha(\tan^2\alpha + 3) T g]. \end{aligned} \tag{24}$$

To obtain further identities, we write

$$\begin{aligned} 0 &= \int_{\mathbb{S}^1} \partial_\theta (g(\omega + \beta_-, \alpha_-))\ d\theta \\ &= \int_{\mathbb{S}^1} \left(\frac{\partial \beta_-}{\partial \theta}\partial_\beta + \frac{\partial \alpha_-}{\partial \theta}\partial_\alpha \right) g(\omega+\beta_-, \alpha_-)\ d\theta \\ &= I_0^\sharp[\partial_\beta g] + \rho \int_{\mathbb{S}^1} \frac{\cos\theta}{\cos\alpha_-} T g(\omega + \beta_-, \alpha_-)\ d\theta, \end{aligned}$$

as well as

$$\begin{aligned} 0 &= \int_{\mathbb{S}^1} \partial_\theta^2 (g(\omega+\beta_-, \alpha_-))\ d\theta \\ &= \int_{\mathbb{S}^1} \partial_\theta \left(\partial_\beta g + \frac{\rho\cos\theta}{\cos\alpha_-} T g \right)\ d\theta \\ &= \int_{\mathbb{S}^1} \left(\partial_\beta^2 g + \frac{2\rho\cos\theta}{\cos\alpha_-} T\partial_\beta g - \left(\frac{\rho\sin\theta}{\cos\alpha_-} + \rho^2\cos^2\theta \frac{\sin\alpha_-}{\cos^3\alpha_-}\right) T g + \frac{\rho^2\cos^2\theta}{\cos^2\alpha_-} T^2 g \right)\ d\theta. \end{aligned}$$

From the previous identity and the fact that $T\partial_\beta = \partial_\beta T$, the second term equals $-2I_0^\sharp[\partial_\beta^2 g]$. In the remaining terms, we use that $-\rho\sin\theta = \sin\alpha_-$ and $\rho^2\cos^2\theta = \rho^2(1-\sin^2\theta) = \rho^2 - \sin^2\alpha_-$ and the previous equality becomes

$$\begin{aligned} \frac{1}{\rho^2} I_0^\sharp[\tan^2\alpha T^2 g - \tan\alpha(1+\tan^2\alpha) T g] &= -\frac{1}{\rho^2} I_0^\sharp[\partial_\beta^2 g] + I_0^\sharp[-\tan\alpha(1+\tan^2\alpha) T g + (1+\tan^2\alpha) T^2 g] \\ &= -\frac{1}{\rho^2}\partial_\omega^2 I_0^\sharp g + I_0^\sharp[-\tan\alpha(1+\tan^2\alpha) T g + (1+\tan^2\alpha) T^2 g]. \end{aligned}$$

Plugging this relation into the right hand side of (24), we obtain

$$\left[(1-\rho^2)\partial_\rho^2 + \left(\frac{1}{\rho} - 3\rho\right)\partial_\rho\right] I_0^\sharp g = -\frac{1}{\rho^2}\partial_\omega^2 I_0^\sharp g + I_0^\sharp[(T^2 + 2\tan\alpha T)g],$$

hence (22) is proved. Equation (23) follows immediately once noticing that

$$D = \frac{1}{\mu}T^2\mu + 1,$$

thus Theorem 3 is proved. □

An integration by parts with zero boundary terms (notice that ρ and $1-\rho^2$ both vanish at the ends of $[0,1]$) shows that for all $u, v \in C^\infty(\mathbb{D})$,

$$(\mathcal{L}u, v)_{L^2(\mathbb{D})} = \int_{\mathbb{D}} \left((1-\rho^2)(\partial_\rho u)\overline{(\partial_\rho v)} + \frac{1}{\rho^2}(\partial_\omega u)\overline{\partial_\omega v}\right)\ \rho\ d\rho\ d\omega + (u,v)_{L^2(\mathbb{D})}, \tag{25}$$

in particular $\mathcal{L}$ is a self-adjoint operator as densely defined on $L^2(\mathbb{D})$ (it is defined on $C^\infty(\mathbb{D})$). In addition, the operator $-T^2$ is formally self-adjoint on $L^2_+(\partial_+ SM)$ or $C^\infty_{\alpha,-,+}(\partial_+ SM)$ defined in Indeed, following notation in [MM19], an orthogonal basis of $L^2_+(\partial_+ S\mathbb{D})$ whose C^∞ span gives $C^\infty_{\alpha,-,+}(\partial_+ SM)$ is given by

$$\psi_{n,k} := \frac{(-1)^n}{4\pi} e^{i(n-2k)(\beta+\alpha)}(e^{i(n+1)\alpha} + (-1)^n e^{-i(n+1)\alpha}), \qquad n \ge 0, \quad k \in \mathbb{Z}, \tag{26}$$

and such that $(-T^2)\psi_{n,k} = (n+1)^2\psi_{n,k}$ for all n, k.

From these observations, passing to the adjoints in (23), the further intertwining property holds

$$I_0 \circ \mathcal{L} = (-T^2) \circ I_0. \tag{27}$$

2.3.4 Backprojecting the joint eigenfunctions of $-T^2$ and $-i\partial_\beta$ - Zernike polynomials

We now focus our attention to $I_0^*\psi_{n,k} = I_0^\sharp\left[\frac{\psi_{n,k}}{\mu}\right]$. Together with the definition of $I_0^\sharp$ and the relations satisfied by the Euclidean footpoint map for all $(\rho e^{i\omega}, \theta) \in S\mathbb{D}$:

$$\begin{aligned}
&\beta_-(\rho e^{i\omega}, \theta) + \alpha_-(\rho e^{i\omega}, \theta) + \pi = \theta, \\
&\beta_-\left(\rho e^{i\omega}, \theta\right) = \beta_-(\rho, \theta-\omega) + \omega, \qquad \alpha_-(\rho e^{i\omega}, \theta) = \alpha_-(\rho, \theta-\omega),
\end{aligned}$$

we arrive at the expression

$$I_0^\sharp\left[\frac{\psi_{n,k}}{\mu}\right](\rho e^{i\omega}) = e^{i(n-2k)\omega}\frac{1}{2\pi}\int_{\mathbb{S}^1} e^{i(n-2k)\theta}\frac{e^{i(n+1)\alpha_-(\rho,\theta)} + (-1)^n e^{-i(n+1)\alpha_-(\rho,\theta)}}{2\cos\alpha_-(\rho,\theta)}\ d\theta.$$

With the relation $\sin\alpha_-(\rho,\theta) = -\rho\sin\theta$, we may rewrite this as

$$I_0^\sharp\left[\frac{\psi_{n,k}}{\mu}\right](\rho e^{i\omega}) = \frac{e^{i(n-2k)\omega}}{2\pi}\int_{\mathbb{S}^1} e^{i(n-2k)\theta} W_n(-\rho\sin\theta)\ d\theta, \tag{28}$$

where we have defined

$$W_n(\sin\alpha) := \frac{e^{i(n+1)\alpha} + (-1)^n e^{-i(n+1)\alpha}}{2\cos\alpha}. \tag{29}$$

The functions W_n are related to the Chebychev polynomials of the second kind U_n, specifically through the relation $W_n(t) = i^n U_n(t)$. In particular, it is immediate to check the 2-step recursion relation and initial conditions

$$W_{n+1}(t) = 2itW_n(t) + W_{n-1}(t), \qquad W_0(t) = 1, \qquad W_1(t) = 2it.$$

By induction, the top-degree term of W_n is $(2it)^n$. Fixing $n \ge 0$, we now split the calculation into two cases:

Case $k < 0$ or $k > n$. In light of (28), since W_n is a polynomial of degree n, then $W_n(-\rho\sin\theta)$ is a trigonometric polynomial of degree n in $e^{i\theta}$. In particular, if $k<0$ or $k>n$, then $|n-2k|>n$ and thus the right hand side of (28) is identically zero. In short, we deduce

$$I_0^\sharp\left[\frac{\psi_{n,k}}{\mu}\right] = 0, \qquad n \ge 0, \qquad k<0 \text{ or } k>n.$$

Case $0 \le k \le n$. For the remaining cases, we then define $Z_{n,k} := I_0^\sharp\left[\frac{\psi_{n,k}}{\cos\alpha}\right]$, and for the sake of self-containment, we now show that the functions $\{Z_{n,k}\}_{n\ge 0,\ 0\le k\le n}$ so constructed are the Zernike basis in the convention of [KB04], by showing that they satisfy Cauchy-Riemann systems and take the same boundary values.

Lemma 4. *The functions $\{Z_{n,k}\}_{n\ge 0,\ 0\le k\le n}$ satisfy the following properties: For all $n \ge 0$*

$$\partial_{\bar z} Z_{n,0} = 0, \qquad \partial_z Z_{n,k} + \partial_{\bar z} Z_{n,k+1} = 0 \qquad (0 \le k \le n-1), \qquad \partial_z Z_{n,n} = 0, \tag{30}$$

$$Z_{n,k}(e^{i\omega}) = (-1)^k e^{i(n-2k)\omega}, \qquad 0 \le k \le n, \quad \omega \in \mathbb{S}^1. \tag{31}$$

Proof. Using the relation $W_n(-t) = (-1)^n W_n(t)$, we arrive at the expression

$$\begin{aligned} Z_{n,k}(\rho e^{i\omega}) &= e^{i(n-2k)\omega}\frac{(-1)^n}{2\pi}\int_{\mathbb{S}^1} e^{i(n-2k)\theta} W_n(\rho\sin\theta)\ d\theta \\ &= \frac{(-1)^n}{2\pi}\int_{\mathbb{S}^1} e^{i(n-2k)\theta} W_n(\rho\sin(\theta-\omega))\ d\theta. \end{aligned} \tag{32}$$

With $\partial_z = \frac{e^{-i\omega}}{2}(\partial_\rho - \frac{i}{\rho}\partial_\omega)$ and $\partial_{\bar z} = \frac{e^{i\omega}}{2}(\partial_\rho + \frac{i}{\rho}\partial_\omega)$, we compute

$$\partial_z(\rho\sin(\theta-\omega)) = i\frac{e^{-i\theta}}{2}, \qquad \partial_{\bar z}(\rho\sin(\theta-\omega)) = -i\frac{e^{i\theta}}{2}.$$

Plugging these into (32) immediately implies

$$\partial_z Z_{n,k} + \partial_{\bar z} Z_{n,k+1} = 0, \qquad 0 \le k \le n-1. \tag{33}$$

In addition, we compute

$$
\begin{aligned}
Z_{n,0}(\rho e^{i\omega}) &= e^{in\omega} \frac{(-1)^n}{2\pi} \int_{\mathbb{S}^1} e^{in\theta} W_n(\rho \sin\theta)\ d\theta \\
&= e^{in\omega} \frac{(-1)^n}{2\pi} \int_{\mathbb{S}^1} e^{in\theta} (2i\rho\sin\theta)^n\ d\theta \\
&= \rho^n e^{in\omega} \frac{(-1)^n}{2\pi} \int_{\mathbb{S}^1} e^{in\theta} (2i\sin\theta)^n\ d\theta
\end{aligned}
$$

where the second equality comes from the fact that the lower-order terms of $W_n(\rho\sin\theta)$ have no harmonic content along $e^{in\theta}$. Finally, the constant is

$$
\int_{\mathbb{S}^1} e^{in\theta} (e^{i\theta} - e^{-i\theta})^n\ d\theta = \int_{\mathbb{S}^1} (e^{2i\theta} - 1)^n\ d\theta = 2\pi(-1)^n.
$$

In short, $Z_{n,0} = \rho^n e^{in\omega} = z^n$. This also implies $\partial_{\overline{z}} Z_{n,0} = 0$ and since we have $Z_{n,n} = (-1)^n \overline{Z_{n,0}} = (-1)^n \overline{z}^n$, we deduce that $\partial_z Z_{n,n} = 0$.

To prove the boundary condition, using that $Z_{n,k}(\rho e^{i\omega}) = e^{i(n-2k)\omega} Z_{n,k}(\rho)$, it is enough to show that $Z_{n,k}(1) = (-1)^k$ for every $n \ge 0$ and $0 \le k \le n$. That this is true for $k = 0$ and $k = n$ follows from the expressions just computed, and the general claim follows by induction on n once the following equality is satisfied:

$$
Z_{n,k}(1) = Z_{n-2,k-1}(1) - Z_{n-1,k-1}(1) + Z_{n-1,k}(1). \tag{34}
$$

To prove (34), it suffices to input the recursion $W_n(\sin\theta) = 2i\sin\theta W_{n-1}(\sin\theta) + W_{n-2}(\sin\theta)$ into the expression (32), and to evaluate it at $\rho e^{i\omega} = 1$. □

From Lemma 4, we see that the family so defined satisfies the characterization (b) of [KB04, Theorem 1] of the Zernike polynomials. One may see that this characterization defines the same family due the following facts: for $n \ge 0$ and $k = 0$, the functions $Z_{n,k}$ in both sets agree; by induction on $k > 0$, in both sets of functions, $Z_{n,k}$ satisfies a $\partial_{\overline{z}}$ equation with same right-hand side and same boundary condition, for which a solution is unique if it exists.

Let us then a few useful properties of these polynomials:

- The following characterization is proved in [KB04, Theorem 1]:

$$
Z_{n,k}(z,\overline{z}) = \frac{1}{k!} \frac{\partial^k}{\partial z^k} \left[z^n \left(\frac{1}{z} - \overline{z} \right)^k \right], \quad n \ge 0, \quad 0 \le k \le n. \tag{35}
$$

- The family $\{Z_{n,k}\}_{n\ge 0, 0\le k\le n}$ is orthogonal on $L^2(\mathbb{D})$. Indeed, they are the eigenfunction of the pair of self-adjoint operators $(\mathcal{L}, -\partial_\omega^2)$ (as densely defined on $C^\infty(\mathbb{D})$), since we have

$$
(\mathcal{L}, -\partial_\omega^2) Z_{n,k} = (\mathcal{L}, -\partial_\omega^2) I_0^* \psi_{n,k} = I_0^*(-T^2, -\partial_\beta^2)\psi_{n,k} = ((n+1)^2, (n-2k)^2) Z_{n,k},
$$

  and the map $(n,k) \mapsto ((n+1)^2, (n-2k)^2)$ is injective.

- Their completeness in $L^2(\mathbb{D})$ follows again from the Weierstrass approximation theorem.

We finally show

Lemma 5.

$$\|Z_{n,k}\|^2 = \frac{\pi}{n+1}, \qquad n \ge 0, \qquad 0 \le k \le n. \tag{36}$$

A functional-analytic proof is given in [Mon20, Appendix]. We give here a proof in the spirit of recurrence relations for orthogonal polynomials.

Proof of Lemma 5. As a quick consequence of (35), the following relation holds

$$(n+1) Z_{n,k} = -\overline{\partial}(Z_{n+1,k+1} + Z_{n-1,k}), \qquad n \ge 0,\ 0 \le k \le n. \tag{37}$$

Multiplying (37) by $\overline{Z_{n,k}}$ and integrating, we arrive at the relation

$$(n+1)\|Z_{n,k}\|^2 = -\int_{\mathbb{D}} (\overline{\partial} Z_{n+1,k+1}) \overline{Z_{n,k}} + \int_{\mathbb{D}} (\overline{\partial} Z_{n-1,k}) \overline{Z_{n,k}}.$$

The last term is zero because $\overline{\partial} Z_{n-1,k}$ is of degree $n-2$ and $Z_{n,k}$ is orthogonal to any polynomial of degree $\ge n-1$. On to the first term,

$$\begin{aligned} \int_{\mathbb{D}} (\overline{\partial} Z_{n+1,k+1}) \overline{Z_{n,k}} &= \int_{\mathbb{D}} \overline{\partial}(Z_{n+1,k+1} \overline{Z_{n,k}}) - \int_{\mathbb{D}} Z_{n+1,k+1} \overline{\partial Z_{n,k}}, \\ &= -\frac{1}{2i} \int_{\partial\mathbb{D}} Z_{n+1,k+1} \overline{Z_{n,k}}\ dz - \int_{\mathbb{D}} Z_{n+1,k+1} \overline{\partial Z_{n,k}}. \end{aligned}$$

The rightmost term is again zero by consideration of degree, while the boundary term is computed using (31), to wit

$$\int_{\mathbb{D}} (\overline{\partial} Z_{n+1,k+1} \overline{Z_{n,k}})\ dz = \frac{-1}{2i} \int_{\mathbb{S}^1} (-1)^{k+1} e^{i(n+1-2(k+1))\beta} e^{-i(n-2k)\beta} (-1)^k i e^{i\beta}\ d\beta = \frac{1}{2} \int_{\mathbb{S}^1} d\beta = \pi,$$

hence (36) is proved. □

Exercise 15. *Prove* (37) *using* (35).

2.3.5 SVD of I_0 and mapping properties

We now conclude regarding the SVD of I_0 using the method of intertwining differential operators. This SVD has been known for quite some time, see e.g. [Cor64, Lou84], and the idea to use intertwining differential operators for such derivations can be found e.g. in [Maa91], though they are usually written there for each polar harmonic number separately.

Equation (23) allows to avoid this separation by harmonics. Below, the "hat" notation stands for vector normalization in their respective spaces.

Theorem 6. *The Singular Value Decomposition of* $I_0 \colon L^2(\mathbb{D}) \to L^2(\partial_+ S\mathbb{D}, d\Sigma^2)$ *is given by*

$$(\widehat{Z_{n,k}}, \widehat{\psi_{n,k}}, a_{n,k})_{n\ge 0, 0\le k\le n}, \qquad a_{n,k} := \frac{\sqrt{4\pi}}{\sqrt{n+1}}. \tag{38}$$

Proof. We obviously have $(-T^2)\psi_{n,k} = (n+1)^2\psi_{n,k}$ and $-i\partial_\beta\psi_{n,k} = (n-2k)\psi_{n,k}$, which by self-adjointness on $L^2(\partial_+ S\mathbb{D}, d\Sigma^2)$ of the two operators applied, makes $\psi_{n,k}$ and orthogonal system. In addition, an immediate computation gives

$$\|\psi_{n,k}\|^2_{L^2(\partial_+ SM)} = \frac{1}{4}, \qquad n\ge 0, \qquad k\in\mathbb{Z}.$$

In addition we have, as explained in [MM19] $I_0^*\psi_{n,k} = 0$ for $k<0$ or $k>n$, and for $0\le k\le n$, we define $Z_{n,k} := I_0^*\psi_{n,k}$. By Theorem 3, we compute

$$\begin{aligned} \mathcal{L}Z_{n,k} &= \mathcal{L}I_0^*\psi_{n,k} = I_0^*(-T^2)\psi_{n,k} = (n+1)^2 Z_{n,k} \\ -i\partial_\omega Z_{n,k} &= (n-2k)Z_{n,k}, \end{aligned}$$

which immediately makes them an orthogonal system in $L^2(\mathbb{D})$. This gives us orthogonal systems associated with I_0 and I_0^* and to compute the singular values, it suffices to normalize all vectors. By definition we have

$$I_0^*\ \widehat{\psi_{n,k}} = a_{n,k}\ \widehat{Z_{n,k}}, \qquad a_{n,k} := \frac{\|Z_{n,k}\|_{L^2(\mathbb{D})}}{\|\psi_{n,k}\|_{L^2(\partial_+ S\mathbb{D})}} = 2\|Z_{n,k}\|_{L^2(\mathbb{D})} \overset{(36)}{=} \frac{\sqrt{4\pi}}{\sqrt{n+1}},$$

hence the result. □

The following statement follows directly. Although it is unclear whether it appears explicitly in the literature, the ingredients for the proof were known since Zernike's seminal paper [Zer34].

Theorem 7. *The following relation holds:*

$$\mathcal{L}(I_0^* I_0)^2 = (4\pi)^2 Id.$$

Proof. The proof is seen at the level of the spectral decomposition, since we have for every $n\ge 0$ and $0\le k\le n$,

$$I_0^* I_0 Z_{n,k} = \frac{4\pi}{n+1} Z_{n,k}, \qquad \text{and} \qquad \mathcal{L}Z_{n,k} = (n+1)^2 Z_{n,k}.$$

□

A Sobolev-Zernike scale. For $s\in\mathbb{R}$, let us define the scale of Hilbert spaces

$$\begin{aligned} \widetilde{H}^s(\mathbb{D}) &= \left\{ f = \sum_{n=0}^\infty \sum_{k=0}^n f_{n,k}\widehat{Z_{n,k}}, \qquad \sum_{n=0}^\infty (n+1)^{2s} \sum_{k=0}^n |f_{n,k}|^2 < \infty \right\} \\ &= \left\{ f\in L^2(\mathbb{D}),\ \mathcal{L}^{s/2} f \in L^2(\mathbb{D}) \right\}, \end{aligned} \tag{39}$$

with continuous, in fact compact, injections $\widetilde{H}^s \subset \widetilde{H}^t$ for $s>t$. An important property of the scale $\{\widetilde{H}^s(\mathbb{D})\}_s$ is the following:

Theorem 8.

$$\bigcap_{s\in\mathbb{R}} \widetilde{H}^s(\mathbb{D}) = C^\infty(\mathbb{D})$$

Proof. The inclusion $\supset$ is clear, since a smooth function f is such that for all $n \ge 0$, $\mathcal{L}^n f \in L^2(\mathbb{D})$. The proof of the inclusion $\subset$ is based on the next two lemmas, proved in [Mon20, Appendix].

Lemma 9. *For all $\alpha > 3/2$, we have the continuous injection $\widetilde{H}^\alpha(\mathbb{D}) \to C(\mathbb{D})$.*

Lemma 10. *There exists $\ell > 0$ such that for every $\alpha \ge \ell$, the operators*

$$\partial\colon \widetilde{H}^\alpha(\mathbb{D}) \to \widetilde{H}^{\alpha-\ell}(\mathbb{D}) \qquad \textit{and} \qquad \overline{\partial}\colon \widetilde{H}^\alpha(\mathbb{D}) \to \widetilde{H}^{\alpha-\ell}(\mathbb{D})$$

are bounded. The index ℓ can be chosen as $2+\varepsilon$ for every $\varepsilon > 0$.

To prove the inclusion $\subset$, it is enough to show that if $f \in \cap_{s\ge 0}\widetilde{H}^s(\mathbb{D})$, then for any $p, q \ge 0$, $\partial^p\overline{\partial}^q f \in C(\mathbb{D})$. With ℓ a constant as in Lemma 10, since $f \in \widetilde{H}^{(p+q)\ell+3}(\mathbb{D})$, repeated use of Lemma 10 gives that $\partial^p\overline{\partial}^q f \in \widetilde{H}^3(\mathbb{D})$, and by Lemma 9, this implies that $\partial^p\overline{\partial}^q f \in C(\mathbb{D})$. Hence the result. $\square$

Moreover, it is immediate to establish the property, for all $s \ge 0$

$$\|I_0^* I_0 f\|_{\widetilde{H}^{s+1}} = 4\pi \|f\|_{\widetilde{H}^s}, \qquad \forall f \in \widetilde{H}^s, \tag{40}$$

which is both a continuity and stability estimate.

We also fully understand the mapping properties of $I_0\colon L^2(M) \to L^2_+(\partial_+ SM)$: if $f = \sum_{n,k} f_{n,k}\widehat{Z_{n,k}}$, then

$$I_0 f = \sum_{n,k} f_{n,k} \frac{\sqrt{4\pi}}{\sqrt{n+1}} \widehat{\psi_{n,k}}.$$

Similarly to above, one may define a scale of spaces

$$\begin{aligned} H^s_{T,+}(\partial_+ SM) &= \left\{ g = \sum_{n=0}^\infty \sum_{k\in\mathbb{Z}} g_{n,k}\widehat{\psi_{n,k}}, \quad \sum_{n=0}^\infty (n+1)^{2s} \sum_{k\in\mathbb{Z}} |g_{n,k}|^2 < \infty \right\} \\ &= \left\{ g \in L^2_+(\partial_+ SM), \quad (-T^2)^{s/2} g \in L^2_+(\partial_+ SM) \right\}, \end{aligned}$$

in which the following identity is immediate:

$$\|If\|_{H^{s+1/2}_{T,+}} = \sqrt{4\pi}\|f\|_{\widetilde{H}^s}, \qquad \forall f \in \widetilde{H}^s. \tag{41}$$

Note that the above space is an example of *anisotropic Sobolev space*, defined in terms of some but not all derivatives (in other words, the definition of $H^s_{T,+}$ says nothing about how smooth a function in it is with respect to ∂_β).

Exercise 16. *Show* (40) *and* (41).

Unlike $I_0^* I_0$ which is surjective from $\widetilde{H}^s$ to $\widetilde{H}^{s+1}$, and although (41) holds, and unlike $I_0^* I_0$ which is surjective, this does not say that I_0 is surjective ((41) is indexed by f and says nothing as to whether If exhausts the codomain).

As we will see later, there exists an operator C_- that is $H^s_{T,+} \to H^s_{T,+}$-continuous for every s, and such that

$$I_0(\widetilde{H}^s(\mathbb{D})) = H^{s+\frac{1}{2}}_{T,+}(S\mathbb{D}) \cap \ker C_-.$$

3 A variety of methods to tackle the Radon/X-ray transform on the plane

3.1 Generalities and mapping properties

Recall the definition, for $f \in \mathscr{S}(\mathbb{R}^2)$:

$$Rf(s,\theta) = \int_{\mathbb{R}} f(s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp)\ dt, \qquad (s,\theta) \in \mathbb{R}\times\mathbb{S}^1.$$

One may show that $R\colon \mathscr{S}(\mathbb{R}^2) \to \mathscr{S}(\mathcal{Z})$ is continuous. However, in spite of the fact that we have defined the "adjoint" of R relative to the $L^2(\mathbb{R}^2)$-$L^2(\mathcal{Z})$ pairing, let us first clarify the following point:

the Radon transform $R\colon L^2(\mathbb{R}^2) \to L^2(\mathcal{Z})$ is **not** bounded.

To see this, consider the function $f(\mathrm{x}) = \frac{1}{\langle \mathrm{x}\rangle^\beta}$ where we denote $\langle \mathrm{x}\rangle := (1+|\mathrm{x}|^2)^{1/2}$. We make the following claims:

- $f \in L^2$ iff $\beta > 1$: indeed,

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}^2} \frac{d\mathrm{x}}{(1+|\mathrm{x}|^2)^\beta} = 2\pi \int_0^\infty \frac{\rho\ d\rho}{(1+\rho^2)^\beta},$$

  and the latter is finite iff $2\beta - 1 > 1$.

- If $\beta > 1$, its Radon transform takes the form

$$Rf(s,\theta) = \int_{\mathbb{R}} \frac{1}{(1+s^2+t^2)^{\beta/2}}\ dt = \frac{1}{\langle s\rangle^{\beta-1}} \int_{\mathbb{R}} \frac{du}{(1+u^2)^{\beta/2}} = \frac{A_\beta}{\langle s\rangle^{\beta-1}},$$

  upon changing variable $t = u\sqrt{1+s^2}$. A_β is a fixed number, finite iff $\beta > 1$.

- For $\beta > 1$, $Rf \in L^2(\mathcal{Z})$ iff $\beta > \frac{3}{2}$: indeed,

$$\|Rf\|_{L^2(\mathcal{Z})}^2 = A_\beta^2 2\pi \int_{\mathbb{R}} \frac{ds}{\langle s\rangle^{2\beta-2}},$$

  and the integral on the right is finite iff $2\beta - 2 > 1$.

As a conclusion for any $1 < \beta \le 3/2$, the function $f(\mathrm{x}) = \langle \mathrm{x}\rangle^{-\beta}$ is in L^2 while Rf is not in L^2.

To reintroduce spaces in which the Radon transform becomes bounded, we need to consider weighted spaces where the weights behave polynomially at infinity. In particular, we define

$$L^2(\mathbb{R}^2, \langle \mathrm{x}\rangle^\alpha) := \{f : \int_{\mathbb{R}^2} |f(\mathrm{x})|^2 \langle \mathrm{x}\rangle^\alpha\ d\mathrm{x} < \infty\},$$

$$L^2(\mathcal{Z}, \langle s\rangle^\alpha) := \{g : \int_{\mathcal{Z}} |g(s,\theta)|^2 \langle s\rangle^\alpha\ ds\ d\theta < \infty\}.$$

With these definitions, we prove the following:

Theorem 11. *For any $\alpha > \frac{1}{2}$, the Radon transform is continuous in the following setting:*

$$R\colon L^2(\mathbb{R}^2, \langle \mathrm{x}\rangle^{2\alpha}) \to L^2(\mathcal{Z}, \langle s\rangle^{2\alpha-1}),$$

with operator norm no greater than $\sqrt{2\pi A_{2\alpha}}$, $A_{2\alpha} := \int_{\mathbb{R}} \frac{dt}{(1+t^2)^\alpha}$.

Proof. Suppose $\alpha > \frac{1}{2}$, then use Cauchy-Schwarz inequality to make appear

$$\begin{aligned} |Rf|(s,\theta)^2 &= \left| \int_{\mathbb{R}} f(s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp) \frac{(1+s^2+t^2)^{\alpha/2}}{(1+s^2+t^2)^{\alpha/2}}\ dt \right| \\ &\le \int_{\mathbb{R}} |f(s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp)|^2 \langle s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp\rangle^\alpha\ dt \cdot \int_{\mathbb{R}} \frac{dt}{(1+s^2+t^2)^\alpha}. \end{aligned}$$

Changing variable $t = \sqrt{1+s^2}u$, we arrive at

$$\int_{\mathbb{R}} \frac{dt}{(1+s^2+t^2)^\alpha} = \frac{A_{2\alpha}}{\langle s\rangle^{2\alpha-1}}, \qquad A_{2\alpha} = \int_{\mathbb{R}} \frac{dt}{(1+t^2)^\alpha} < \infty.$$

Multiplying through by $\langle s\rangle^{2\alpha-1}$, we arrive at

$$|Rf|(s,\theta)^2 \langle s\rangle^{2\alpha-1} \le A_{2\alpha} \int_{\mathbb{R}} |f(s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp)|^2 \langle s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp\rangle^\alpha\ dt.$$

Now integrate w.r.t. $ds\ d\theta$, change variable $\mathrm{x}(s,t) = s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp$ in the R.H.S. to arrive at the estimate

$$\|Rf\|^2_{L^2(\mathcal{Z}, \langle s\rangle^{2\alpha-1})} \le 2\pi A_{2\alpha} \|f\|^2_{L^2(\mathbb{R}^2, \langle \mathrm{x}\rangle^{2\alpha})}.$$

Hence the result. □

Exercise 17. *Find an expression for A_β in terms of the Beta function*

$$B(x,y) := 2\int_0^{\pi/2} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1}\ d\theta, \qquad x > 0,\ y > 0.$$

Remark 2. *A special case of Theorem 11 is for $\alpha = \frac{1}{2}$, where the operator*

$$R\colon L^2(\mathbb{R}^2, \langle \mathrm{x}\rangle) \to L^2(\mathcal{Z})$$

is bounded, with adjoint $R^* := \langle \mathrm{x}\rangle^{-1} R^t$, if R^t denotes the unweighted $L^2 - L^2$-adjoint previously defined.

The necessity of moment conditions The operator $R\colon \mathscr{S}(\mathbb{R}^2) \to \mathscr{S}(\mathcal{Z})$ is not surjective. To see this, we have the following:

Lemma 12. *If g is in the range of $R\colon \mathscr{S}(\mathbb{R}^2) \to \mathscr{S}(\mathcal{Z})$, then for every $k \in \mathbb{N}_0$, $\int_{\mathbb{R}} g(s,\theta) s^k\ ds$ is a homogeneous polynomial of degree k in $\cos\theta, \sin\theta$.*

Proof. Suppose $g = Rf$ for some $f \in \mathscr{S}(\mathbb{R}^2)$ and let $k \ge 0$. We compute

$$
\begin{aligned}
\int_{\mathbb{R}} s^k Rf(s,\theta)\ ds &= \int_{\mathbb{R}^2} s^k f(s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp)\ dt\ ds \\
&= \int_{\mathbb{R}^2} (\mathrm{x}\cdot\boldsymbol{\theta})^k f(\mathrm{x})\ dx \qquad (\text{setting } \mathrm{x}(s,t) = s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp) \\
&= \sum_{j=0}^k \binom{k}{j} \cos^j\theta \sin^{k-j}\theta \int_{\mathbb{R}^2} f(\mathrm{x}) x^j y^{k-j}\ d\mathrm{x},
\end{aligned}
$$

hence the result. □

For example, $g(s,\theta) = e^{-s^2} e^{i\theta}$ cannot be in the range of R, since $\int_{\mathbb{R}} g(s,\theta)\ ds = \sqrt{2\pi} e^{i\theta}$ is not a polynomial of degree zero in $\cos\theta, \sin\theta$.

It can be shown that the converse also holds, namely that if these conditions are satisfied, then $g = Rf$ for some f. In fact, such an f can be uniquely constructed from the moments $\int_{\mathbb{R}^2} f(\mathrm{x}) x^j y^{k-j}\ d\mathrm{x}$ appear above.

3.2 The Fourier Slice Theorem and its consequences

3.2.1 The Fourier Slice Theorem

We first describe our Hilbert scales on $\mathbb{R}^2$ and $\mathcal{Z}$. We define the Fourier transform on $\mathbb{R}^2$ as usual:

$$
\hat{f}(\xi) = \int_{\mathbb{R}^2} e^{-i\mathrm{x}\cdot\xi} f(\mathrm{x})\ d\mathrm{x}, \qquad \xi \in \mathbb{R}^2, \qquad f \in L^2(\mathbb{R}^2),
$$

and on $\mathcal{Z}$,

$$
\tilde{g}(\sigma,\theta) = \int_{\mathbb{R}} e^{-is\sigma} g(s,\theta)\ ds, \qquad g \in L^2(\mathcal{Z}).
$$

for $r \in \mathbb{R}$, define

$$
\|f\|^2_{H^r}(\mathbb{R}^2) = \int_{\mathbb{R}^2} (1+|\xi|^2)^r |\hat{f}(\xi)|^2\ d\xi, \qquad \|g\|^2_{H^s(\mathcal{Z})} = \int_{\mathcal{Z}} (1+\sigma^2)^r |\tilde{g}(\sigma,\theta)|^2\ d\sigma\ d\theta.
$$

Theorem 13 (Fourier Slice Theorem)**.** *For all $f \in \mathscr{S}(\mathbb{R}^2)$,*

$$
\widetilde{Rf}(\sigma,\theta) = \hat{f}(\sigma\boldsymbol{\theta}), \qquad (\sigma,\theta) \in \mathcal{Z}.
$$

Proof.

$$
\widetilde{Rf}(\sigma,\theta) = \int_{\mathbb{R}} e^{-i\sigma s} Rf(s,\theta)\ ds = \int_{\mathbb{R}} e^{-i\sigma s} \int_{\mathbb{R}} f(s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp)\ dt\ ds.
$$

The result follows by changing variable $\mathrm{x}(s,t) = s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp$, noticing that $s\sigma = \mathrm{x}\cdot\boldsymbol{\theta}\sigma = \mathrm{x}\cdot(\sigma\boldsymbol{\theta})$. □

Some consequences:

- If $Rf = 0$, then $f = 0$ (Injectivity # 2).
- The FST motivates a first reconstruction formula: take Rf, compute its 1D Fourier transform in s, this is the Fourier transform of f in polar coordinates; recover f from its Fourier transform. This last step may require a tricky interpolation from polar to cartesian, and we will see below other reconstruction formulas which do not have this drawback.

3.2.2 Stability estimates

In the Hilbert scales defined above, the FST allows to formulate sharp stability estimates.

Theorem 14 (two-sided estimates, see Thm 2.2.2. in [Bal12])**.** *Let $f \in H^r(\mathbb{R}^n)$ for some $r \in \mathbb{R}$. Then we have the following inequalities:*

(i) $\sqrt{2}\|f\|_{H^r(\mathbb{R}^2)} \le \|Rf\|_{H^{r+1/2}(\mathcal{Z})}$.

(ii) For any smooth and compactly supported function χ,

$$\|R(\chi f)\|_{H^{r+1/2}(\mathcal{Z})} \le C_\chi \|\chi f\|_{H^r(\mathbb{R}^2)}.$$

In particular the estimates can not be improved on the Sobolev scale, the problem is "ill-posed of order 1/2".

Proof. For the purpose of proving both (i) and (ii), we first compute, as a preliminary:

$$\|Rf\|^2_{H^{r+\frac{1}{2}}(\mathcal{Z})} = \int_{\mathbb{R}\times\mathbb{S}^1} |\widetilde{Rf}(\sigma,\theta)|^2 \langle\sigma\rangle^{2r+1}\, d\sigma\, d\theta = 2\int_{(0,\infty)\times\mathbb{S}^1} |\widetilde{Rf}(\sigma,\theta)|^2 \langle\sigma\rangle^{2r+1}\, d\sigma\, d\theta,$$

where we have used the symmetry $\widetilde{Rf}(\sigma,\theta) = \widetilde{Rf}(-\sigma,\theta+\pi)$. We now use the Fourier Slice Theorem and change variable $\xi = \sigma\boldsymbol{\theta}$ to arrive at

$$\|Rf\|^2_{H^{r+\frac{1}{2}}(\mathcal{Z})} = 2\int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \langle\xi\rangle^{2r} \frac{\langle\xi\rangle}{|\xi|}\, d\xi.$$

Now notice that we *always* have $\langle\xi\rangle \ge |\xi|$, so estimate (i) following immediately from bounding $\frac{\langle\xi\rangle}{|\xi|}$ from below by 1.

To obtain (ii) however, we see that there is an issue[17] because $\frac{\langle\xi\rangle}{|\xi|}$ is not bounded on $\mathbb{R}^2$. Fixing a "cutoff" function $\chi \in C_c^\infty(\mathbb{R}^2)$, we rewrite the last equation as

$$\frac{1}{2}\|R(\chi f)\|^2_{H^{r+\frac{1}{2}}(\mathcal{Z})} = \int_{\mathbb{R}^2} |\widehat{\chi f}(\xi)|^2 \langle\xi\rangle^{2r} \frac{\langle\xi\rangle}{|\xi|}\, d\xi = \underbrace{\int_{|\xi|<1}}_{I_1} + \underbrace{\int_{|\xi|\ge 1}}_{I_2}.$$

[17]This is sometimes referred to as a zero-frequency problem, since it occurs at $\xi = 0$.

On the term I_2, for $|\xi| \ge 1$, we always have $\frac{\langle\xi\rangle}{|\xi|} = \left(\frac{1}{|\xi|^2}+1\right)^{1/2} \le \sqrt{2}$, and hence $I_2 \le \sqrt{2}\|\chi f\|_{H^r}$. Bounding I_1 is the trickier part. We exploit the fact that the singularity at $\xi = 0$ is integrable:

$$I_1 = \int_{|\xi|\le 1} |\widehat{\chi f}(\xi)|^2 \langle\xi\rangle^{2r} \frac{\langle\xi\rangle}{|\xi|}\ d\xi \le \underbrace{\int_{|\xi|\le 1} \langle\xi\rangle^{2r} \frac{\langle\xi\rangle}{|\xi|}\ d\xi}_{=C<\infty} \cdot \sup_{|\xi|\le 1} |\widehat{\chi f}(\xi)|.$$

Now fix $\psi \in C_c^\infty(\mathbb{R}^2)$ equal to 1 on the support of χ so that $\chi f = \psi\chi f$. Then for $|\xi| \le 1$,

$$\begin{aligned}
\widehat{\chi f}(\xi) = \widehat{\chi f \psi}(\xi) &= \int_{\mathbb{R}^2} \chi(\mathrm{x}) f(\mathrm{x}) \psi(\mathrm{x}) e^{-i\mathrm{x}\cdot\xi}\ d\mathrm{x} \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{\chi f}(\eta)\ \widehat{\psi e^{-i(\cdot)\cdot\xi}}(\eta)\ d\eta \qquad \text{(Parseval)} \\
&\le \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\widehat{\chi f}(\eta)|^2 \langle\eta\rangle^{2r}\ d\eta \cdot \int_{\mathbb{R}^2} |\widehat{\psi e^{-i(\cdot)\cdot\xi}}(\eta)|^2 \langle\eta\rangle^{-2r}\ d\eta \qquad \text{(Cauchy-Schwarz)} \\
&\le \frac{1}{(2\pi)^2} \|\chi f\|_{H^r}^2 \int_{\mathbb{R}^2} |\widehat{\psi}(\eta+\xi)|^2 \langle\eta\rangle^{-2r}\ d\eta.
\end{aligned}$$

The last factor is uniformly bounded over the set $|\xi| \le 1$. Putting everything back together,

$$I_1 \le C_\chi \|\chi f\|_{H^r},$$

where the constant C_χ may depend on the size of the support of f (through the choice of function ψ). $\square$

3.2.3 Interlude: Riesz potentials, Hilbert transform and convolution operators

This section serves as preliminary for the one that follows.

Recall the identity $\widehat{f\star g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$. It's true for $f, g \in \mathscr{S}(\mathbb{R}^n)$ but can be pushed to $g \in \mathscr{S}'$ and f a distribution with compact support.

Given f a distribution with compact support[18], one may consider the convolution operator $A_f \colon \mathscr{S} \to \mathscr{S}'$ defined by $A_f(g) := f \star g$. The previous identity tells us that it can be computed in two ways: (i) by direct computation of the convolution, or (ii) via Fourier transform, as $A_f = \mathcal{F}^{-1} \circ \hat{f}(\xi) \circ \mathcal{F}$. In other context, the function $\hat{f}(\xi)$ is called the *symbol* (or Fourier multiplier) of A_f, and A_f is called the *quantization* of $\hat{f}$. The form (ii) is extremely useful for computing purposes, as the computation of $\mathcal{F}$ and $\mathcal{F}^{-1}$ can be done using the *Fast Fourier Fransform* (FFT).

Several operators are convolution operators in disguise:

- The identity is nothing but $g \mapsto \delta \star g$, which, as a Fourier multiplier, gives $\hat{f}(\xi) = 1$. In particular, $\hat{\delta} = 1$.
- Any differential operator $P(D) = \sum_{|\alpha|\le m} a_\alpha \partial^\alpha$ can be viewed as convolution by $P(D)\delta$, with Fourier multilplier $P(i\xi)$. In particular, the Laplacian Δ has symbol $-|\xi|^2$.

A class of interest to us in the next section will be the **Riesz potentials**: on $\mathbb{R}^n$, for $\alpha < n$, define I^α via Fourier transform:

$$\widehat{I^\alpha f}(\xi) := |\xi|^{-\alpha} \hat{f}(\xi). \tag{42}$$

The condition $\alpha < n$ ensures that $|\xi|^{-\alpha}\hat{f}(\xi)$ remains locally integrable so that it can be viewed as an element of $\mathscr{S}'$.

Some comments: for $\alpha < 0$, I^α is "unsmoothing", for example, $I^{-2} = -\Delta$; for $\alpha = 0$, I^0 is the identity; for $0 < \alpha < n$, I^α is smoothing. It is also immediately clear that $I^\alpha \circ I^\beta = I^\beta \circ I^\alpha = I^{\alpha+\beta}$ as long as α, β and $\alpha + \beta$ are strictly less than n.

The Hilbert transform. Picking up where we left off, let us focus on the one-dimensional case, using s for the physical variable and σ for its dual Fourier variable. The operator I^{-1} is then a convolution operator with symbol $|\sigma|$, and one may wonder how close it is from being a $\frac{d}{ds}$ derivative, an operator whose symbol is $i\sigma$. The answer is fairly simple: upon writing

$$|\sigma| = i\sigma \cdot \frac{1}{i}\mathrm{sgn}(\sigma),$$

we see that I^{-1} can be written as the product of two commuting operators, one being $\frac{d}{ds}$, and the other one being the operator with Fourier multiplier $\hat{h}(\sigma) := \frac{1}{i}\mathrm{sgn}(\sigma)$. We call that operator the **Hilbert transform**. Namely, define

$$H \colon L^2(\mathbb{R}) \to L^2(\mathbb{R}), \qquad Hf := \mathcal{F}^{-1}(\hat{h}(\xi)\mathcal{F}f). \tag{43}$$

[18] An element of $\mathscr{E}'$ in the PDE notes.

From this definition, several interesting properties follow quickly: Using Parseval's formula, H is an isometry of L^2; moreover, $H^2 = -Id$.

One may however wonder what that operator looks like as a convolution operator. Namely: if $\hat{h}(\sigma) := \frac{1}{i}\text{sgn}(\sigma)$, what does $h \in \mathscr{S}'$ look like ? Upon defining the distribution $p.v.\frac{1}{s}$ by

$$\langle p.v.\frac{1}{s}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} := \lim_{\varepsilon\to 0} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{\varphi(s)}{s}\ ds = \int_0^\infty \frac{\varphi(s)-\varphi(-s)}{s}\ ds,$$

We now sketch a proof that

$$h(s) = \frac{1}{\pi} p.v.\frac{1}{s}. \tag{44}$$

Proof of (44). We first show that $\frac{d}{d\sigma}\text{sgn}(\sigma) = 2\delta$. Indeed, for any $\varphi \in \mathscr{S}$,

$$\begin{aligned}\langle \frac{d}{d\sigma}\text{sgn}(\sigma), \varphi\rangle = -\langle \text{sgn}(\sigma), \varphi'\rangle = -\int_{\mathbb{R}} \text{sgn}(\sigma)\varphi'(\sigma)\ d\sigma \\ = \int_{-\infty}^0 \varphi'(\sigma)\ d\sigma + \int_0^\infty \varphi'(\sigma)\ d\sigma = 2\varphi(0) = \langle 2\delta, \varphi\rangle.\end{aligned}$$

In addition, we claim that $\delta = \widehat{\frac{1}{2\pi}}$, as the line below shows:

$$\langle \delta, \varphi\rangle = \varphi(0) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\varphi}(\xi)\ d\xi = \langle \frac{1}{2\pi}, \hat{\varphi}\rangle = \langle \widehat{\frac{1}{2\pi}}, \varphi\rangle.$$

Combining the past two claims, we arrive at the conclusion that

$$\widehat{-ish}(\sigma) = \frac{d}{d\sigma}\hat{h}(\sigma) = \frac{d}{d\sigma}\frac{1}{i}\text{sgn}(\sigma) = \frac{2}{i}\delta = \frac{2}{i}\widehat{\frac{1}{2\pi}}.$$

Since the Fourier transform is an isomorphism, we deduce the equality of tempered distributions:

$$sh(s) = \frac{1}{\pi}. \tag{45}$$

One may be tempted to divide by s and call it a day, but the function $\frac{1}{s}$ does not define a distribution. However, the distribution $\frac{1}{\pi}p.v.\frac{1}{s}$ does, and in fact, solves equation (45) (exercise: check this). As a result, we have the following equality of tempered distributions

$$s\left(h(s) - \frac{1}{\pi}p.v.\frac{1}{s}\right) = 0.$$

We now use the following lemma, whose sketch is given in Ex. 18:

Lemma 15. *If $u \in \mathscr{S}'$ solves $su = 0$ in $\mathscr{S}'$, then $u = C\delta$ for some constant C.*

We then deduce that

$$h(s) = \frac{1}{\pi}p.v.\frac{1}{s} + C\delta.$$

Finally, C is zero by evenness/oddness considerations: h and $\frac{1}{\pi}p.v.\frac{1}{s}$ are both odd in the sense that $\langle h, \varphi(-s)\rangle = -\langle h, \varphi\rangle$ for all $\varphi \in \mathscr{S}$, and δ is even in the sense that $\langle \delta, \varphi(-s)\rangle = \langle \delta, \varphi\rangle$ for all $\varphi \in \mathscr{S}$. The last display in the equality forces C to be zero. Hence the result. □

What equation (44) buys us is the following second characterization of the Hilbert transform:

$$Hf(t) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(s)}{t-s}\, ds, \tag{46}$$

which we will use later.

Exercise 18. *Sketch of proof of Lemma 15: suppose $su = 0$. Fix $\chi \in C_c^\infty(\mathbb{R})$ a function equal to 1 in a neighbourhood of 0. For $\varphi \in \mathscr{S}$, write $\varphi(s) = \varphi(0)\chi(s) + s\psi(s)$ for some $\psi \in \mathscr{S}$. Conclude by computing $\langle u, \varphi\rangle$ using this decomposition.*

3.2.4 Filtered-Backprojection formulas

Exact formulas. In what follows, we will somewhat abuse notation: for a function on $\mathbb{R}^2$, we will define the Riesz potential $I^\alpha f$ as in (42). For functions on $\mathcal{Z}$, we will use the 1D Riesz potential w.r.t. the s variable, namely

$$\widetilde{I^\alpha g}(\sigma, \theta) := |\sigma|^{-\alpha} \tilde{g}(\sigma, \theta).$$

Using these operators, we can then derive the following one-parameter family of inversion formulas, see also [Nat01, Thm. 2.1]. Recall the definition of the backprojection operator

$$R^t g(\mathrm{x}) = \int_{\mathbb{S}^1} g(\mathrm{x}\cdot\boldsymbol{\theta}, \theta)\, d\theta, \qquad \mathrm{x} \in \mathbb{R}^2.$$

Theorem 16. *For all $\alpha < 2$ and $f \in \mathscr{S}(\mathbb{R}^2)$,*

$$f(x) = \frac{1}{4\pi} I^{-\alpha} R^t I^{\alpha-1} R f.$$

Proof. We compute, using the Fourier inversion formula

$$\begin{aligned}
I^\alpha f(\mathrm{x}) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{I^\alpha f}(\xi) e^{i\mathrm{x}\cdot\xi}\, d\xi \\
&\overset{(a)}{=} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\xi|^{-\alpha} \hat{f}(\xi) e^{i\mathrm{x}\cdot\xi}\, d\xi \\
&= \frac{1}{(2\pi)^2} \int_0^\infty \int_{\mathbb{S}^1} |\sigma|^{1-\alpha} \hat{f}(\sigma\boldsymbol{\theta}) e^{i\mathrm{x}\cdot\sigma\boldsymbol{\theta}}\, d\sigma\, d\theta \qquad (\xi = \sigma\boldsymbol{\theta}) \\
&\overset{(b)}{=} \frac{1}{(2\pi)^2} \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{S}^1} |\sigma|^{1-\alpha} \widetilde{Rf}(\sigma, \theta) e^{i\mathrm{x}\cdot\sigma\boldsymbol{\theta}}\, d\sigma\, d\theta \\
&= \frac{1}{4\pi} \int_{\mathbb{S}^1} I^{1-\alpha} Rf(\mathrm{x}\cdot\boldsymbol{\theta}, \theta)\, d\theta \\
&= \frac{1}{4\pi} R^t I^{1-\alpha} Rf(\mathrm{x}).
\end{aligned}$$

In (a) we have used the definition of I^α, and in (b) we have used the symmetry $\widetilde{Rf}(-\sigma, \theta+\pi) = \widetilde{Rf}(\sigma, \theta)$ to extend the integral to $\mathbb{R}$. ☐

Special cases:

- The case $\alpha = 1$ reads

$$f = \frac{1}{4\pi} I^{-1} R^t R f,$$

where I^{-1} corresponds to the Fourier multiplier $|\xi|$, i.e. $I^{-1} = \sqrt{-\Delta}$. We could have guessed this from the work that was done in Section 2.2.

- Perhaps the most popular case is when $\alpha = 0$, this gives:

$$f = \frac{1}{4\pi} R^t I^{-1} R f. \tag{47}$$

The operator I^{-1} is the one-dimensional Fourier multiplier $|\sigma|$. Computing it is done columnwise (for each θ separately), and each column is processed via `fast fourier transform` before and after multiplication of by $|\sigma|$. As explained above we can write $I^{-1} = \frac{d}{ds} H = H \frac{d}{ds}$.

- The last comment allows one to easily go from (47) to Radon's original inversion formula:

$$f(\mathrm{x}) = \frac{-1}{\pi} \int_0^\infty \frac{dF_{\mathrm{x}}(q)}{q}, \qquad F_{\mathrm{x}}(q) := \frac{1}{2\pi} \int_{\mathbb{S}^1} Rf(q + \mathrm{x} \cdot \boldsymbol{\theta}, \theta)\ d\theta.$$

This formula has a nice geometric interpretation: $f(\mathrm{x})$ is a weighted functional of $F_{\mathrm{x}}(q)$, which is the average of Rf over all lines tangent to the circle of center x and radius q.

Approximate formulas.

The Fourier multiplier $|\sigma|$, also called a *ramp filter*, amplifies high frequencies more than low frequencies. Since actual images have limited bandwidth[19], and since noise tends to be more prominent at high frequencies, we want to replace this "exact" filter by a low-pass one, emphasizing the features of f which can be more faithfully reconstructed by the data at hand.

The general setting is the following: take $w \in \mathscr{S}(\mathbb{R})$, $w \equiv w(s)$ an even function (in the sense that $w(-s) = w(s)$), and let $W = R^t w$, that is to say

$$W(\mathrm{x}) = \int_{\mathbb{S}^1} w(\mathrm{x} \cdot \boldsymbol{\theta})\ d\theta = W(|\mathrm{x}|).$$

Then following the same scheme of proof as Theorem 16, one can show the following general result. In the statement $\overset{\mathrm{x}}{\star}$ denotes two-dimensional convolution and $\overset{s}{\star}$ denotes one-dimensional s-convolution.

Theorem 17 (Filtered-Backprojection Formulas)**.**

$$W \overset{\mathrm{x}}{\star} f = \frac{1}{4\pi} R^t (w \overset{s}{\star} Rf). \tag{48}$$

[19] i.e., their Fourier transforms is supported in a ball of radius b. The smallest such b is often called the bandwidth.

Formulas of this type do not reconstruct f exactly, rather, they reconstruct $W \star f$, which should be thought of as a "regularized", low-frequency version of the unknown.

The relevant parameter is $\tilde{w}(\sigma)$, which should be in some sense a low-pass version of $|\sigma|$. Fixing a cutoff bandwidth $b > 0$, here are the most commonly used filters in Matlab's `iradon` function.

`Ram-Lak`: $\tilde{w}_b(\sigma) = |\sigma|\chi_{[-b,b]}(\sigma)$.

`Shepp-Logan`: $\tilde{w}_b(\sigma) = |\sigma|\chi_{[-b,b]}(\sigma)\frac{\sin(\pi\sigma/2b)}{\pi\sigma/2b}$.

`cosine`: $\tilde{w}_b(\sigma) = |\sigma|\chi_{[-b,b]}(\sigma)\cos(\frac{\pi}{2}\frac{|\sigma|}{b})$.

`Hahn`: $\tilde{w}_b(\sigma) = |\sigma|\chi_{[-b,b]}(\sigma)(.54 + .46\cos(\pi\sigma/b))$.

`Hamming`: $\tilde{w}_b(\sigma) = |\sigma|\chi_{[-b,b]}(\sigma)(.5 + .5\cos(\pi\sigma/b))$.

Approximate formulas via functional calculus and intertwining differential operators. We take a second look at Theorem 17. Since the function w is only a function of s, it is easy to show that $W = R^t w$ is only a function of $|\mathrm{x}|$. This implies that $\widehat{W}(\xi)$ is only a function of $|\xi|$, or equivalently, of $|\xi|^2$, namely, let us write $\widehat{W}(\xi) = \widehat{W}_r(|\xi|^2)$ for some function $\widehat{W}_r\colon (0,\infty) \to \mathbb{R}$.

The LHS of (48) looks like $W \star f = \mathcal{F}^{-1} \circ \widehat{W}_r(|\xi|^2) \circ \mathcal{F} f$, something we can write $\widehat{W}_r(-\Delta) f$. Such an identification is obviously justified whenever $\widehat{W}_r$ is a polynomial, and for more general functions, it is justified by the *spectral theorem for self-adjoint operators.*

In short, for a general bounded function $h \in L^\infty(0,\infty)$, we can make sense of $h(-\Delta)$ via the formula $\widehat{h(-\Delta)f}(\xi) = h(|\xi|^2)\hat{f}(\xi)$, and for functions on $\mathcal{Z}$, we can make sense of $h(-\partial_s^2)$ via the formula $\widetilde{h(-\partial_s^2)g}(\sigma,\theta) = h(|\sigma|^2)\tilde{g}(\sigma,\theta)$ (recall that $\tilde{g}(\sigma,\theta) = \int_{\mathbb{R}} e^{-is\theta} g(s,\theta)\ ds$).

The important thing to notice next is that since the relation $R \circ (-\Delta) = (-\partial_s^2) \circ R$ generalizes to any reasonable function of $-\Delta$, namely $R \circ h(-\Delta) = h(-\partial_s^2) \circ R$. Again, such a formula is obvious when h is a polynomial, and can be generalized to any limit of polynomials.

For such a function, apply the formula (47) to $h(-\Delta)f$ to make appear

$$h(-\Delta)f = \frac{1}{4\pi} R^t H \partial_s R(h(-\Delta)f) = \frac{1}{4\pi} R^t H \partial_s h(-\partial_s^2) R f. \tag{49}$$

This formula tells us what to do in order to recover a regularized version of f, $h(-\Delta)f$, from Rf. The operator $H\partial_s h(-\partial_s^2)$ has symbol $|\sigma| h(|\sigma|^2)$.

Examples of regularization kernels. In the last paragraph above, by 'regularized', we mean a low-frequency version of the function to be reconstructed. In particular, the function $h(\rho)$ should decay to zero as $\rho \to \infty$. The `ram-lak` filter above corresponds to the sharp cutoff

$$h(\rho) = 1_{[0,b^2]}(\rho^2) \qquad (\texttt{ram-lak}),$$

where b is the cutoff bandwidth. At the level of the reconstruction, this amounts to using the filter $\widehat{W}(\xi) = 1_{[0,b^2]}(|\xi|^2)$, which is nothing but the characteristic function of the ball of radius b. Thus the function $h(-\Delta)f$ only contains the frequency content of f up to frequency b. This induces a limitation on the size of the details of $h(-\Delta)f$.

Another example of regularization kernel is given by the *heat semi-group*

$$h(-\Delta) = e^{t\Delta} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\Delta^k, \qquad t > 0.$$

(it's a semi-group because $e^{t\Delta} \circ e^{s\Delta} = e^{(t+s)\Delta}$ for all $s, t > 0$). It is both a non-polynomial example of a relevant function of the Laplacian, and its existence is motivated by PDEs. Specifically: for $f \in \mathscr{S}'(\mathbb{R}^2)$, $e^{t\Delta} f$ is obtained by solving the following heat equation:

$$\frac{\partial u}{\partial t} = \Delta u \qquad (t > 0,\ \mathrm{x} \in \mathbb{R}^2) \quad \text{ with initial condition } \qquad u|_{t=0} = f.$$

Indeed, we can look at what the function $v(\xi, t) := \int_{\mathbb{R}^2} e^{i\mathrm{x}\cdot\xi} u(\mathrm{x}, t)\ d\mathrm{x}$ satisfies: using the properties of the Fourier transform, v satisfies $\frac{dv}{dt} = -|\xi|^2 v$ with initial condition $v(\xi, 0) = \hat{f}(\xi)$, and this can be immediately solved through direct ODE integration

$$v(\xi, t) = e^{-t|\xi|^2}\hat{f}(\xi).$$

We then see that $u(\mathrm{x}, t) = e^{t\Delta} f$ as claimed. Note that this also means that $u(\cdot, t)$ can be viewed as a convolution operator applied to f, with convolution kernel

$$W_t(\mathrm{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\mathrm{x}\cdot\xi} e^{-t|\xi|^2}\ dt = \frac{1}{4\pi t} e^{-|\mathrm{x}|^2/4t}.$$

Exercise 19. *Prove Theorem 17.*

Exercise 20. *For $(\alpha, \mathbf{a}) \in \mathbb{S}^1 \times \mathbb{R}^2$, define the action of the Euclidean group*

$$(\alpha, \mathbf{a}) \cdot f(\mathrm{x}) = f(R(\alpha)\mathrm{x} + \mathbf{a}), \qquad R(\alpha) := \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}, \qquad f \in \mathscr{S}(\mathbb{R}^2).$$

Find a relation between $R[(\alpha, \mathbf{a}) \cdot f]$ and Rf.

Exercise 21. *The Radon transform in higher dimensions. For $f \in \mathcal{S}(\mathbb{R}^n)$, define*

$$Rf(s, \omega) = \int_{\{x\cdot\omega = s\}} f, \qquad s \in \mathbb{R},\ \omega \in \mathbb{S}^{n-1},$$

where the hyperplane $h_{s,\omega} := \{x \in \mathbb{R}^n,\ x \cdot \omega = s\}$ is equipped with its natural Lebesgue measure. Notice the symmetry $Rf(s, \omega) = Rf(-s, -\omega)$. Define $\hat{f}$ the Fourier transform as usual on $\mathbb{R}^n$, and $\tilde{g}(\sigma, \omega)$ the one-dimensional Fourier transform along the s factor for functions on $\mathbb{R} \times \mathbb{S}^{n-1}$.

1. *Prove the Fourier slice theorem: for any $f \in \mathcal{S}(\mathbb{R}^n)$, $\widetilde{Rf}(\sigma, \omega) = \hat{f}(\sigma\omega)$,*
2. *Compute the formal adjoint operator of $R : L^2(\mathbb{R}^n) \to L^2(\mathbb{R} \times \mathbb{S}^{n-1})$, call it R^*. What is its geometric meaning ?*
3. *Recalling the definition of the Riesz potentials as usual (for $\alpha < n$, $\widehat{I^\alpha f}(\xi) = |\xi|^{-\alpha}\hat{f}(\xi)$), prove the filtered-backprojection formula, true for any $f \in \mathcal{S}(\mathbb{R}^n)$:*

$$f = \frac{1}{2}(2\pi)^{1-n} I^{-\alpha} R^* I^{\alpha-n+1} Rf, \qquad \alpha < n.$$

4. *Focus on the case $\alpha = 0$ and explain how the locality of I^{-n+1} depends on the parity of n.*
5. *Conclude that in odd dimensions, $f(x)$ can be reconstructed from the Radon transform over all planes intersecting a small neighborhood of x.*

Exercise 22. *Radon transform and wave equation. Recall d'Alembert's formula*

$$v(x, t) = \frac{1}{2}(f(x - t) + f(x + t)) + \frac{1}{2}\int_{x-t}^{x+t} g(u)\ du$$

which provides the expression for the unique solution to the $1 + 1$-dimensional wave problem

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} = 0 \qquad (x \in \mathbb{R},\ t > 0), \qquad v|_{t=0} = f, \qquad \frac{\partial v}{\partial t}|_{t=0} = g.$$

This problem shows how the Radon transform and d'Alembert's formula provide a method for solving wave equations in $\mathbb{R}^n \times (0, \infty)$ for any $n \in \mathbb{N}$.

1. *Prove that for $f \in \mathcal{S}(\mathbb{R}^n)$, $\frac{d^2}{ds^2} Rf = R[\Delta f]$.*
2. *Use the previous result to derive a solution of the wave problem*

$$\partial_{tt} u - \Delta u = 0 \qquad (x \in \mathbb{R}^n, t > 0), \qquad u(x, 0) = u_0(x), \qquad \partial_t u(x, 0) = u_1(x),$$

   by setting up a PDE problem for the function $v(s, \omega, t) := Ru(x, t)$ (where the Radon transform acts on the x-variable only).

3.3 Diagonalization through circular harmonics and radial inversions

3.3.1 Preliminaries on the Beta and Gamma functions

Recall that we define the Gamma function via the absolutely convergent integral $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\, dt$ for all $s > 0$. This defines a smooth function which interpolates the factorial in the sense that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}_0$.

Exercise 23. *1. Show that $\Gamma(s+1) = s\Gamma(s)$ for all $s > 0$. Deduce by induction that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}_0$.*

2. Show that $\Gamma(1/2) = \sqrt{\pi}$. Deduce the value of Γ at all half-integers.

Also define the Beta function for all $a, b > 0$ by

$$B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\; du. \tag{50}$$

It is also smooth in both arguments, and can in fact be expressed in terms of the Gamma function as follows:

$$B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\; du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{51}$$

It contains the value of all binomial coefficients, Wallis integrals, etc.

Exercise 24. *Prove (51) as follows: write $\Gamma(a)\Gamma(b)$ as a double integral in $(t,u) \in \mathbb{R}^2_+$ and do the change of variable $v = t+u$, $w = \frac{t}{t+u}$.*

3.3.2 The Radon transform of radial functions

Suppose a smooth function to be integrated takes the form $f(x) = F(|x|)$ for F a compactly supported function. Then by direct calculation,

$$\begin{aligned} Rf(s,\theta) &= \int_{\mathbb{R}} F(|s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp|)\; dt \\ &= 2\int_0^\infty F(\sqrt{s^2+t^2})\; dt \\ &= 2\int_s^\infty \frac{F(u)u}{\sqrt{u^2-s^2}}\; du =: R_0F(s). \end{aligned}$$

We immediately see that Rf only depends on s, and that the recovery of F leads to the inversion of the one-dimensional integral operator $F \mapsto R_0F$. To understand the nature of this operator, let us take a detour through a family of integral transforms of Abel type.

Exercise 25. *Show that if $f \in C^\infty(\mathbb{R}^2)$ and $f(\mathrm{x}) = F(|\mathrm{x}|)$ for some $F : [0,\infty) \to \mathbb{R}$, then F must take the form $F(\rho) = g(\rho^2)$ for some function $g \in C^\infty([0,\infty))$. [Hint: if f is smooth at 0, write a Taylor expansion there.]*

Abel transforms. In the sequel, all the calculations can be justified by considering functions which are continuous and compactly supported on $\mathbb{R}_+$, a space which we write $C_c(\mathbb{R}_+)$. If you are curious, think about how far one can reduce the regularity and integrability below and still make the calculations justified.

Following [Eps08, §2.5.4], for $0 < \alpha \le 1$, we define the Abel transform

$$A_\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty \frac{g(s)}{(s-t)^{1-\alpha}}\, ds, \qquad g \in C_c(\mathbb{R}_+).$$

with, in particular when $\alpha = 1/2$:

$$A_{1/2} g(t) = \frac{1}{\sqrt{\pi}} \int_t^\infty \frac{g(s)}{\sqrt{s-t}}\, ds, \qquad g \in C_c(\mathbb{R}_+).$$

Notice that A_1 produces an antiderivative for g, so these operators have a smoothing behavior. In fact, A_α is smoothing by the fractional amount α. This is best embodied by the following observation:

Lemma 18. *For all $\alpha \in (0,1)$,*

$$A_\alpha \circ A_{1-\alpha} = A_{1-\alpha} \circ A_\alpha = A_1 \qquad \text{on } C_c(\mathbb{R}_+).$$

Proof. We need to show that $A_{1-\alpha} A_\alpha h(t) = \int_t^\infty h(u)\, du$. Iterating the integral, we get

$$\begin{aligned} A_{1-\alpha} A_\alpha h(t) &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_t^\infty \frac{1}{(s-t)^\alpha} \int_s^\infty \frac{h(u)}{(u-s)^{1-\alpha}}\, du\, ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_t^\infty h(u) \left(\int_t^u \frac{ds}{(u-s)^{1-\alpha}(s-t)^\alpha} \right) du. \end{aligned}$$

Now, somewhat magically, upon changing variable in the s integral $s = t + v(u-t)$ for $v \in [0,1]$, we arrive at

$$\int_t^u \frac{ds}{(u-s)^{1-\alpha}(s-t)^\alpha} = \int_0^1 \frac{dv}{(1-v)^{1-\alpha} v^\alpha} = B(\alpha, 1-\alpha) = \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)},$$

and the result follows. $\square$

In fact, since A_1 is inverted by the operator $-\partial_x$, Lemma 18 gives us an inverse for A_α: we directly have

$$-\partial_x \circ A_{1-\alpha} \circ A_\alpha = id. \qquad \text{(inversion of } A_\alpha) \tag{52}$$

Exercise 26. *Show the generalization of Lemma 18: as long as $\alpha, \beta, \alpha+\beta \in (0,1)$, $A_\alpha \circ A_\beta = A_{\alpha+\beta}$.*

Inversion of R_0. We now use the results from the previous paragraph to invert R_0 explicitly. This is done by first relating R_0 with the Abel transform $A_{1/2}$. For a function $F\colon [0,\infty) \to \mathbb{R}$, denote $F_{\surd}(t) := F(\sqrt{t})$. Working on the transform R_0:

$$R_0 F(s) = 2\int_s^\infty \frac{F(u)u}{\sqrt{u^2-s^2}}\ du = \int_s^\infty \frac{F_{\surd}(u^2)}{\sqrt{u^2-s^2}} d(u^2) = \int_{s^2}^\infty \frac{F_{\surd}(t)}{\sqrt{t-s^2}}\ dt = A_{1/2} F_{\surd}(s^2).$$

In other words,

$$(R_0 F)\circ \sqrt{\cdot} = \sqrt{\pi} A_{1/2}(F\circ\sqrt{\cdot}), \qquad F\in C_c(\mathbb{R}_+), \tag{53}$$

i.e. R_0 is nothing but $A_{1/2}$ acting on functions deformed by the squareroot function.

Since the inversion of $A_{1/2}$ follows from (52) with $\alpha = 1/2$, one is able to easily deduce the inversion of R_0:

Theorem 19 (Inversion of R_0)**.**

$$F(r) = \frac{-1}{\pi r}\partial_r \left[\int_r^\infty \frac{R_0 F(s) s}{\sqrt{s^2-r^2}}\ ds\right] = \frac{-1}{2\pi r}\partial_r R_0[R_0 F].$$

Exercise 27. *Prove Theorem 19 using* (53) *and* (52)*.*

An interesting consequence: if $R_0 F$ is supported in $s\le L$, then the form of the reconstruction formula in Theorem 19 tells us that F is also supported in $\{|\mathrm{x}|\le L\}$. This is a first example of the so-called Helgason support theorem, which tells us how we can "scrape off" the support of a function based on the support of its Radon transform.

3.3.3 Diagonalization through harmonics

This material is in part inspired by [Ilm17, Sec. 4, 5].

Let us now generalize to functions which are not necessarily radial. Given $f\in C_c(\mathbb{R}^2)$, for every $\rho\ge 0$, the function $\beta\mapsto f(\rho\boldsymbol{\beta}) = f(\rho\cos\beta, \rho\sin\beta)$ is 2π-periodic and therefore decomposes in Fourier series:

$$f(\rho\boldsymbol{\beta}) = \sum_{k\in\mathbb{Z}} f_k(\rho) e^{ik\beta}, \qquad f_k(\rho) := \frac{1}{2\pi}\int_{\mathbb{S}^1} f(\rho\boldsymbol{\beta}) e^{-ik\beta}\ d\beta. \tag{54}$$

Similarly, the function $Rf(s,\theta)$ is 2π-periodic in θ and thus decomposes into

$$Rf(s,\theta) = \sum_{p\in\mathbb{Z}} (Rf)_p(s) e^{ip\theta}, \qquad (Rf)_p(s) := \frac{1}{2\pi}\int_{\mathbb{S}^1} Rf(s,\theta) e^{-ip\theta}\ d\theta.$$

In all generality, $(Rf)_p(s)$ should depend on all Fourier modes of $f_k(\rho)$, by means of some doubly infinite family of operators $\mathcal{R}_{pk}\colon C_c(\mathbb{R}_+)\to C_c(\mathbb{R}_+)$ via the relation $(Rf)_p = \sum_{k\in\mathbb{Z}} \mathcal{R}_{pk} f_k$. However, all the non-diagonal ones will be zero, because the Radon transform commutes with rotations.

This is seen more directly by simply computing the Radon transform of the k-th term in (54):

$$R[f_k(\rho)e^{ik\beta}](s,\theta) = \int_{\mathbb{R}} f_k(|s\boldsymbol{\theta} + t\boldsymbol{\theta}^\perp|)e^{ik\arg(s\boldsymbol{\theta}+t\boldsymbol{\theta}^\perp)}\ dt$$
$$= \int_{\mathbb{R}} f_k(\sqrt{s^2+t^2})e^{ik\arg(s\boldsymbol{\theta}+t\boldsymbol{\theta}^\perp)}\ dt$$

Now, if $s>0$, $\arg(s\boldsymbol{\theta}+t\boldsymbol{\theta}^\perp) = \theta+\arg(s+it) = \theta+\arctan(t/s) = \theta+\arccos(s/\sqrt{t^2+s^2})$. Returning to our computation:

$$\begin{aligned} R[f_k(\rho)e^{ik\beta}](s,\theta) &= e^{ik\theta}\int_{\mathbb{R}} f_k(\sqrt{s^2+t^2})e^{ik\arccos(s/\sqrt{t^2+s^2})}\ dt \\ &\stackrel{a}{=} e^{ik\theta}\int_0^\infty f_k(\sqrt{s^2+t^2})2\cos(k\arccos(s/\sqrt{t^2+s^2}))\ dt \\ &\stackrel{b}{=} e^{ik\theta}\int_s^\infty f_k(u)\cos(k\arccos(s/u))\ \frac{2udu}{\sqrt{u^2-s^2}} \\ &\stackrel{c}{=} e^{ik\theta}\int_s^\infty f_k(u)T_k(s/u)\ \frac{2udu}{\sqrt{u^2-s^2}}, \end{aligned}$$

where $T_k(x) := \cos(k\arccos x)$ is the k-th Chebyshev polynomial of the first kind[20].

Exercise 28. *Justify steps a, b, c above.*

Exercise 29. *Define the function T_k through the relation $T_k(\cos\theta) = \cos(k\theta)$, $\theta\in[0,\pi]$.*

1. *Show that $T_0(x) = 1$ and $T_1(x) = x$.*
2. *Show the recursion relation $T_{k+2}(x) = 2xT_{k+1}(x) - T_k(x)$. Deduce that T_k is a polynomial of degree k (in particular it is also defined and smooth for x outside $[0,1]$; in fact, on $[1,\infty)$ another interpretation is $T_k(x) = \cosh(k\,\text{arccosh}\ x)$).*

From the previous calculation, we immediately see that $R[f_k(\rho)e^{ik\beta}](s,\theta)$ only contains a Fourier term along $e^{ik\theta}$ and nothing along any other $e^{i\ell\theta}$ for $\ell\neq k$. Moreover, upon defining the Abel type integral transform

$$R_k g(s) := \int_s^\infty g(u)T_k(s/u)\ \frac{2udu}{\sqrt{u^2-s^2}}, \qquad g\in C_c(\mathbb{R}_+). \tag{55}$$

Then we deduce that

$$Rf(s,\theta) = \sum_{k\in\mathbb{Z}} R_k f_k(s)e^{ik\theta},$$

where the f_k terms are the polar Fourier coefficients of f and R_k is defined in (55).

This provides another avenue to reconstruct f from Rf, by solving countably many one-dimensional integral equations, reconstructing each f_k from R_kf_k, provided that this last step is possible. That this is indeed the case is given in the following result.

[20]Really, T_k is initially defined for $k\in\mathbb{N}_0$, but from its defining property, it is clear that we can abuse notation and set $T_k(x) = T_{-k}(x) = T_{|k|}(x)$.

Theorem 20 (Inversion of R_k)**.** *The operator $R_k \colon C_c(\mathbb{R}_+) \to C_c(\mathbb{R}_+)$ is invertible. In particular, $g \in C_c(\mathbb{R}_+)$ can be reconstructed from $R_k g$ by means of the reconstruction formula*

$$g(r) = \frac{1}{\pi} \frac{d}{dr} \int_r^\infty R_k g(s) \frac{T_k(s/r)}{s\sqrt{(s/r)^2 - 1}}\ ds.$$

Remark 3. *Notice that the inversion involves the operator $h \mapsto \int_r^\infty h(s) \frac{T_k(s/r)}{s\sqrt{(s/r)^2-1}}\ ds$, where $T_k(s/r)$ involves values of T_k for $x > 1$ (while the operator involved in the definition of R_k involves values of T_k on $[0,1]$).*

Proof. It is enough to show that

$$\int_r^\infty R_k g(s) \frac{T_k(s/r)}{s\sqrt{(s/r)^2 - 1}}\ ds = -\pi \int_r^\infty g(t)\ dt.$$

Upon writing the double integral:

$$\begin{aligned} \int_r^\infty R_k g(s) \frac{T_k(s/r)}{s\sqrt{(s/r)^2 - 1}}\ ds &= \int_r^\infty \int_s^\infty g(t) T_k(s/t) T_k(s/r) \frac{2t}{\sqrt{t^2 - s^2}} \frac{1}{s\sqrt{(s/r)^2 - 1}}\ dt\ ds \\ &= 2 \int_r^\infty g(t) K_k(r,t)\ dt, \end{aligned}$$

where we have defined

$$K_k(r,t) := \int_r^t \frac{T_k(s/t)}{\sqrt{1 - (s/t)^2}} \frac{T_k(s/r)}{s\sqrt{(s/r)^2 - 1}}\ ds.$$

The result follows upon proving the rather striking fact that $K_k(r,t) = \frac{\pi}{2}$ for all $0 < r < t$ and $k \in \mathbb{N}_0$, which is the object of Exercise 30. $\square$

Exercise 30. *Exercises 52 through 56 + Bonus Exercise 2 in [Ilmavirta, p27] take you through the derivation of the magical identity*

$$\int_r^t \frac{T_k(s/t)}{\sqrt{1 - (s/t)^2}} \frac{T_k(s/r)}{s\sqrt{(s/r)^2 - 1}}\ ds = \frac{\pi}{2}, \qquad 0 < r < t,$$

A consequence of Theorem 20 is as follows:

Corollary 21. *Suppose $f \in C_c(\mathbb{R}^2)$ is such that there is $L \ge 0$ such that $Rf(s,\theta) = 0$ for all $s > L$. Then $f(\mathrm{x}) = 0$ for all $|\mathrm{x}| > L$.*

Proof. Expanding $Rf(s,\theta)$ into Fourier harmonics w.r.t. θ, we obtain that $R_k f_k(s) = 0$ for every $s > L$ and every $k \in \mathbb{Z}$. From the inversion formula in Theorem 20, since $f_k(r)$ is directly reconstructed from $R_k f_k(s)$ for $s \ge r$, this gives that $f_k(r) = 0$ whenever $r \ge L$ for all k, and hence $f(r\boldsymbol{\beta}) = \sum_{k\in\mathbb{Z}} f_k(r) e^{ik\beta} = 0$ whenever $r \ge L$. $\square$

3.3.4 Approximate inversion and Volterra integral equations

The exact inversion of the transform (55) may have looked like it was pulled out of a hat, in the sense that the inverse was given to us. Some of the intuition behind the construction of the inverse may be found in Cormack's original paper [Cor64]. If the kernel seems too messy to intuit an inverse, some general form remains amenable to analysis. Specifically, it is worth noting that all the operators R_k take the form

$$Bf(s) = \int_s^\infty f(u)b(s,u)\frac{2u\ du}{\sqrt{u^2-s^2}}, \qquad s\ge 0, \tag{56}$$

where b is uniformly bounded by 1 (in fact smooth) on $\{(u,s)\in\mathbb{R}_+\times\mathbb{R}_+,\ u>s\}$ and where $b(s,s)=1$ for all $s\ge 0$, and let's assume that we can invert the case where $b(s,u)=1$ for $u\ge s$, in which case the operator B is R_0, the Radon transform on radial functions.

Reduction to a Volterra Integral Equation of the second kind. Assuming that the original integrand f is supported in $[0,L]$, we then see that $Bf(s)=0$ for any $s\ge L$, and thus we may consider the restriction of B to functions on $[0,L]$, defined by

$$Bf(s) = \int_s^L f(u)b(s,u)\frac{2u\ du}{\sqrt{u^2-s^2}}, \qquad s\in[0,L]. \tag{57}$$

and suppose we want to reconstruct f from $g=Bf$, in other words solve $Bf=g$ for f.

From what we've seen when looking at Abel operators, the relative order of smoothing α that A_α provides really comes from the singularity of the kernel at $s=u$. In (57), the only singular behavior comes from $\frac{1}{\sqrt{u-s}}$ which suggests that its smoothing behavior is close to that of the Abel operator $A_{1/2}$ or the operator R_0. Upon plugging $b(s,u)=1+(b(s,u)-1)$ into (57), the equation $Bf=g$ becomes

$$R_0 f + Ef = g, \qquad Ef(s) := \int_s^L f(u)(b(s,u)-1)\frac{2u\ du}{\sqrt{u^2-s^2}}.$$

We then apply the inversion formula of R_0 to make appear

$$f - Kf = -\frac{1}{\pi r}\partial_r R_0 g, \qquad Kf := \frac{1}{\pi r}\partial_r R_0[Ef]. \tag{58}$$

A bit of work shows that:

Lemma 22. *The operator K defined in (58) takes the form $Kf(r)=\int_r^L f(u)k(r,u)\ du$ for some kernel k bounded on $[0,L]^2$.*

Proof. We compute

$$\begin{aligned} R_0Ef(r) &= \int_r^L\int_s^L f(u)(b(s,u)-1)\frac{2u\ du}{\sqrt{u^2-s^2}}\frac{2s\ ds}{\sqrt{s^2-r^2}} \\ &= \int_r^L f(u)\underbrace{\left(\int_r^u (b(s,u)-1)\frac{1}{\sqrt{u^2-s^2}}\frac{2s\ ds}{\sqrt{s^2-r^2}}\right)}_{e(u,r)} 2u\ du. \end{aligned}$$

Now note that since $b(s,s) = 1$, we can write $b(s,u) - 1 = (u^2 - s^2)h(s,u)$ for some bounded, continuous function h. This allows us to write $e(u,r)$ as

$$e(u,r) = \int_r^u h(s,u)\sqrt{u^2 - s^2}\frac{2s\ ds}{\sqrt{s^2 - r^2}},$$

and upon changing variable $s^2 = r^2 + y(u^2 - r^2)$, we arrive at

$$e(u,r) = (u^2 - r^2)\int_0^1 h(\sqrt{r^2 + y(u^2 - r^2)}, u)\frac{\sqrt{1-y}}{\sqrt{y}}dy.$$

In particular, $e(r,r) = 0$ for all r. Finally,

$$\begin{aligned} Kf(r) &= \frac{1}{\pi r}\partial_r R_0 Ef \\ &= \frac{1}{\pi r}\left(2rf(r)e(r,r) + \int_r^L f(u)\partial_r e(u,r)\ 2u\ du\right) \\ &= \frac{1}{\pi r}\int_r^L f(u)\partial_r e(u,r)\ 2u\ du. \end{aligned}$$

We see that K is thus an integral operator with kernel

$$k(r,u) = \frac{2u}{\pi r}\partial_r e(u,r).$$

The lemma is concluded once one shows that k is bounded on $[0,L]$. This is left as an exercise. □

Exercise 31. *Finish the proof of Lemma 22 by showing that the kernel k is uniformly bounded on $[0,L]$.*

With this lemma and the work of the next paragraph, equation (58) can be inverted via the Neumann series solution

$$f = \sum_{p=0}^\infty K^p\left[-\frac{1}{\pi r}\partial_r R_0 g\right],$$

Thereby implying that f can indeed be recovered from g in a semi-explicit manner (modulo an infinite sum !). This method applies to several cases beyond the case of integration along lines: it generalizes to more general integration curves, provided that

- This family has rotational summetry with respect to a distinguished point o.
- Each curve $\gamma(t)$ satisfies $\frac{d^2}{dt^2}|\gamma(t) - o| > 0$, and at the "vertex" $\gamma(t_0)$ where $|\gamma(t_0) - o|$ is minimal, $\gamma(t)$ has first order of contact with the circle to which it is tangent.

Example of articles dealing with such families are: the study of the geodesic X-ray transform for spherically symmetric metrics [Sha97], integration certain kinds of ellipses [AK15] and "Cormack-type curves" [RLL14].

Exercise 32. *What about families of curves with higher order of contact ?*

Volterra operators of the first kind. Some of this material is inspired from [Eps08, §2.5.4].

Define an operator $Kf(r) = \int_r^L f(u)k(r,u)\ du$ for $r \in [0,L]$, where the kernel function $k(r,u)$ satisfies a uniform estimate $|k(r,u)| \le M$ for all $r,u \in [0,L]$. The operator K is called a *Volterra integral operator of the first kind*, and with such an operator, an equation of the form

$$f - Kf = h \qquad \text{on} \qquad [0,L], \tag{59}$$

to be solved for f, is called a *Volterra equation of the second kind.* See [Wid16] for a general treatment. We show the following

Theorem 23. *Let K an operator as above, with kernel $k(r,u)$ uniformly bounded by M on $[0,L]^2$ and vanishing for $u < r$. Then for any $h \in L^\infty([0,L])$, equation (59) has a unique solution $f \in L^\infty([0,L])$ of the form $f = h + \sum_{k=1}^\infty K^p h$, where the second summand is a continuous function.*

Particular cases include:

- if h is continuous on $[0,L]$, then so is f.
- if $h = 0$ on $[0,L]$, then $f = 0$ on $[0,L]$.

Proof. Let us inspect the convergence properties of the series $\sum_{j=0}^\infty K^j h$. Bounding the $j=1$ term, we obtain

$$|Kh(r)| = \left| \int_r^L k(r,u)h(u)\ du \right| \le M\|h\|_\infty (L-r).$$

Notice also that, similarly,

$$|Kh(r) - Kh(r')| \le M\|h\|_\infty |r-r'|,$$

and thus Kh is (Lipschitz-) continuous on $[0,L]$, in particular bounded on $[0,L]$. By induction, every term $K^j h$ will be continuous and bounded. Moreover, using the same bounding technique, one may prove by induction that

$$|K^j h(r)| \le \frac{M^j (L-r)^j}{j!} \|h\|_\infty, \qquad r \in [0,L],$$

this yields the uniform estimate $\|K^j\|_\infty \le \frac{(ML)^j}{j!}\|h\|_\infty$. Since the series of upper bounds is summable, the Weierstrass M-test ensures that the limit exists and is bounded, with sup-norm

$$\left\| \sum_{j=0}^\infty K^j h \right\|_\infty \le e^{ML}\|h\|_\infty.$$

Since all terms but possibly the first one are continuous, then $\sum_{j=1}^\infty K^j h$ is continuous as a uniform limit of continuous functions. $\square$

3.3.5 The exterior problem and Helgason's support theorem

The geometry of lines is such that if $K \subset \mathbb{R}^2$ is compact and convex set, the problem of reconstructing f from Rf admits a natural 'triangular' structure. Namely, if we write $f = f|_K + f|_{K^c}$ (where each restriction is extended by zero to $\mathbb{R}^2$) and set Z_K the set of lines which intersect K, then an obvious observation is that the Radon transform of $f|_K$ is zero on any line that does not intersect K. Hence, the operator R splits into

$$\begin{bmatrix} Rf|_{Z_K} \\ Rf|_{Z_K^c} \end{bmatrix} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \begin{bmatrix} f|_K \\ f|_{K^c} \end{bmatrix}.$$

As for any triangular system, it is then worth inferring first the reconstructibility of $f|_{K^c}$ from $Rf|_{Z_K^c}$. This is called:

The exterior problem: given K a compact and convex domain and $f \in C_c^\infty(\mathbb{R}^2)$,
(i) does $Rf(L) = 0$ for every L not intersecting K imply that $f \equiv 0$ outside K ?
(ii) If the answer to (i) is 'yes', how to recover $f|_{K^c}$ from $Rf|_{Z_K^c}$?

We will address (i), and this is called the Helgason support theorem, whose proof follows from the work done in the previous sections. Recall that the **support** of a function (or a distribution) f is the complement of the largest open set on which $f \equiv 0$.

Theorem 24 (Helgason support theorem)**.** *Let $K \subset \mathbb{R}^2$ be a compact and convex set. Suppose $f \in C_c(\mathbb{R}^2)$ integrates to zero over all lines $L \subset \mathbb{R}^2$ for which $L \cap K = \emptyset$. Then $f|_{\mathbb{R}^2 \setminus K} = 0$.*

Proof. **Case 1: a disk.** Suppose $K = D_r(\mathrm{x}_0)$. Upon considering the function $\tilde{f}(\mathrm{x}) = f(\mathrm{x} - \mathrm{x}_0)$, the goal is to show that if $R\tilde{f}(s,\theta) = 0$ for $s > r$, then $\tilde{f}(\mathrm{x}) = 0$ for $|\mathrm{x}| > r$. But this follows directly from Corollary 21.

Case 2: general. Now that this is proved for a general disk, this can be proved provided that for any compact and convex set K, we have

$$K = \bigcap_{D \supseteq K, D \text{ disk}} D. \tag{60}$$

The inclusion $\subset$ is obvious, and to prove the converse, one must show that for every point $x \notin K$, there is a disk D such that $K \subset D$ and $x \notin D$.

Once (60) is proved: for every disk containing K, Rf vanishes at lines not passing through D, so by case 1, f vanishes on the complement of D. Then f vanishes on $\bigcup D^c = (\bigcap D)^c \overset{(60)}{=} K^c$, which is what we had to prove. $\square$

Exercise 33. *Prove the $\supset$ part of (60).*

Remark 4. *Note that although the exterior problem is injective, and unlike K is at most a point, this problem is severely ill-posed.*

3.4 Approach by complexification and Riemann-Hilbert problem

Further elaboration on this material can be found in [Bal12, Sec. 2.4].

3.4.1 The transport viewpoint

In this section, we take the transport viewpoint to define the X-ray/Radon transform on the plane: given $f \in C_c^\infty(\mathbb{R}^2)$, we define the X-ray transform of f as $If(\mathrm{x}, \theta) := \lim_{t\to\infty} u(\mathrm{x} + t\boldsymbol{\theta}, \boldsymbol{\theta})$, where u is the unique solution to the equation

$$\boldsymbol{\theta} \cdot \nabla u = f \qquad (\mathbb{R}^2 \times \mathbb{S}^1), \qquad \lim_{t\to -\infty} u(\mathrm{x} - t\boldsymbol{\theta}, \boldsymbol{\theta}) = 0.$$

Originally we can think of it as a function of $(\mathrm{x}, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1$, though we see that $If(\mathrm{x} + u\theta, \theta) = If(\mathrm{x}, \theta)$ for any $u \in \mathbb{R}$. Writing $\mathrm{x} = (\mathrm{x}\cdot\boldsymbol{\theta})\boldsymbol{\theta} + (\mathrm{x}\cdot\boldsymbol{\theta}^\perp)\boldsymbol{\theta}^\perp$, we see that $If(\mathrm{x}, \theta) = Rf(\mathrm{x}\cdot\boldsymbol{\theta}^\perp, \theta + \frac{\pi}{2})$, where Rf is the Radon transform in the previous sections. In this convention, integration is done along $\boldsymbol{\theta}$ instead of $\boldsymbol{\theta}^\perp$.

Note that $\mathbb{R}^2 \times \mathbb{S}^1$ can be thought of as the *unit tangent bundle* of $\mathbb{R}^2$, and the equation above can be thought of as a transport equation posed there (in this viewpoint, even the integrand f can be viewed as a function of θ also). The vector field whose integral curves are those over which f is being integrated over is $\boldsymbol{\theta}\cdot\nabla = \cos\theta\partial_x + \sin\theta\partial_y$. It is the *geodesic vector field* on Euclidean $\mathbb{R}^2$, whose integral curves (in $\mathbb{R}^2 \times \mathbb{S}^1$, projected back onto $\mathbb{R}^2$) are straight lines.

3.4.2 Complex preliminaries

We introduce complex coordinates $z = x + iy$, $\overline{z} = x - iy$, and the induced complex derivatives

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \qquad \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

In these coordinates, the area form looks like $dxdy = \frac{d\overline{z}\wedge dz}{2i} =: d\mu(z)$.

- Green's formula in complex coordinates. $\int_{\partial\Omega} Pdx + Qdy = \int_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dx\, dy$ yields

$$\int_{\partial\Omega} u\ dz = 2i \int_\Omega \frac{\partial u}{\partial \overline{z}}\ d\mu(z)$$

- The fundamental solution to $\partial_{\overline{z}} u = f$ with $\lim_{|z|\to\infty} u(z) = 0$ for $f \in C_c(\mathbb{R}^2)$.

$$u(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\zeta)}{z - \zeta}\ d\mu(\zeta).$$

- Plemelj-Sokhotsky formula: For all $f \in C_c^\infty(\mathbb{R})$,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} \frac{f(x)}{ix + \varepsilon}\ dx = -ip.v.\int_{\mathbb{R}} \frac{f(x)}{x}\ dx + \mathrm{sign}(\varepsilon)\pi f(0). \tag{61}$$

3.4.3 A example of a Riemann-Hilbert problem

Call T the unit circle in $\mathbb{R}^2$ traversed counterclockwise, and let D_+ the open unit disk, and D_- the complement of the closed unit disk. If ϕ is smooth on $D_+ \cup D_-$ and has limits at T, we let $\phi_\pm(t) = \lim_{\varepsilon\to 0^+} \phi((1\mp\varepsilon)t)$, and call $\psi(t) := \phi_+(t) - \phi_-(t)$ the jump of ϕ across T.

The RHP associated to T and φ is formulated as follows:

Problem 1 (Riemann-Hilbert problem across T)**.** *Find ϕ satisfying:*

(i) ϕ is analytic on $D_+ \cup D_-$

(ii) $\lambda\phi(\lambda)$ is bounded as $|\lambda| \to \infty$.

(iii) The jump of ϕ across T is ψ.

In their full generality, Riemann-Hilbert problems may include more general jump contours than T, matrix-valued integrands, other normalization conditions than (ii). They appear in several fields of mathematics related to special functions, random matrix theory, orthogonal polynomials, integrable PDEs and more. See [AFF03], and [TO15] for more numerical and practical aspects.

As far as the problem above goes the following result holds and is crucial for what follows.

Theorem 25. *The solution to Problem 1 exists and is unique, given by*

$$\phi(\lambda) = \frac{1}{2\pi i}\int_T \frac{\psi(t)}{t-\lambda}\, dt \qquad \lambda \notin T.$$

Proof. (Existence) That the solution written above solves the problem follows immediately from Plemelj's formula (61), or rather, an adaptation of it to the unit circle. We can also check the jump condition by computing left and right limits and expanding $\psi(e^{i\theta})$ into a Fourier series. Specifically, let us decompose $\psi(e^{i\theta}) = \sum_{k\in\mathbb{Z}} a_k e^{ik\theta}$, and write

$$\phi(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\psi(e^{i\theta})}{e^{i\theta}-\lambda} e^{i\theta}\, d\theta.$$

If $|\lambda| < 1$ we use the Neumann series $\frac{1}{1-\lambda e^{-i\theta}} = \sum_{p\ge 0} \lambda^p e^{-ip\theta}$ to rewrite $\phi(\lambda)$ as

$$\phi(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} \sum_{p\ge 0} e^{-ip\theta}\lambda^p \psi(e^{i\theta})\, d\theta = \sum_{p\ge 0} a_p \lambda^p.$$

As $\lambda \to e^{i\alpha} \in T$, this converges to $\phi_+(e^{i\alpha}) = \sum_{p=0}^\infty a_p e^{ip\alpha}$, and the convergence is justified by the original regularity imposed on $\psi(e^{i\theta})$. If $|\lambda| > 1$, we use the Neumann series $\frac{1}{e^{i\theta}-\lambda} = \frac{1}{\lambda}\sum_{p=0}^\infty \lambda^{-p} e^{ip\theta}$ to write

$$\phi(\lambda) = -\sum_{p\ge 0} a_{-(p+1)} \lambda^{-(p+1)}.$$

As $\lambda \to e^{i\alpha} \in T$, this converges to $\phi_-(e^{i\alpha}) = -\sum_{k=-\infty}^{-1} a_k e^{ik\alpha}$. We indeed see that $\phi_+(e^{i\alpha}) - \phi_-(e^{i\alpha})$ recovers exactly $\psi(e^{i\alpha})$.

(Uniqueness) That the solution is unique follows from Liouville and Morera's theorems: if two solutions ϕ_1, ϕ_2 both solve Problem 1, then $\Phi_1 - \Phi_2$ is entire and bounded near infinity: it must be a constant. Since its limit as $|z| \to \infty$ is zero, it is identically zero. □

3.4.4 Inversion of the X-ray transform through a Riemann Hilbert problem

The idea of this method of inversion is as follows: $u(\mathrm{x}, \theta)$ can be extended to $\lambda \notin T \cup \{0\}$ by complexifying the transport equation, starting from the observation that $\theta \cdot \nabla = e^{i\theta}\partial_z + e^{-i\theta}\partial_{\bar z}$ and setting $\lambda = e^{i\theta}$ and $e^{-i\theta} = \lambda^{-1}$, the augmented function $u(\mathrm{x}, \lambda)$ solves the problem

$$(\lambda\partial_z + \lambda^{-1}\partial_{\bar z})u(z, \lambda) = f(z), \qquad \lim_{|z|\to\infty} u(z, \lambda) = 0, \tag{62}$$

(here we write $u(z)$ but we are not implying that u is complex-analytic). The method of resolution then goes as follows:

1. Freezing $\lambda \notin T \cup \{0\}$, equation (62) turns out to be a complex Cauchy-Riemann equation in the variable $\zeta = \lambda^{-1}z - \lambda\bar z$, in the sense that $\lambda\partial_z + \lambda^{-1}\partial_{\bar z} = (|\lambda|^{-2} - |\lambda|^2)\partial_{\bar\zeta}$. Using the fundamental solution of this problem, we arrive at the expression (denote $u_\pm = u|_{D_\pm}$)

$$u_\pm(z, \lambda) = \int_{\mathbb{C}} G_\pm(z - z', \lambda) f(z')\ d\mu(z'), \qquad G_\pm(z, \lambda) = \frac{\mp 1}{\pi(\lambda^{-1}z - \lambda\bar z)}.$$

2. From the expression above, now freeze z and study $u(z, \lambda)$ for $\lambda \notin T \cup \{0\}$: $u_\pm$ are obviously complex-analytic on their domains of definition, and u_+ is in fact analytic all the way down to $\lambda = 0$, with limit 0. u_- also has the right decay as $|\lambda| \to \infty$, so $u(z, \cdot)$ satisfies a RHP across the unit circle. By Theorem 25, $u(z, \cdot)$ can be globally expressed as a Cauchy integral in terms of its jump across the unit circle. Namely, we also arrive at

$$u(z, \lambda) = \frac{1}{2\pi i}\int_T \frac{\psi(z, t)}{t - \lambda}\ dt = \frac{1}{2\pi}\int_{\mathbb{S}^1} \frac{\psi(z, e^{i\theta})}{1 - \lambda e^{-i\theta}}\ d\theta,$$

where $\psi(z, e^{i\theta}) := \lim_{\varepsilon\to 0^+}\left(u_+(z, (1-\varepsilon)e^{i\theta}) - u_-(z, (1+\varepsilon)e^{i\theta})\right)$ is the jump of u across the unit circle. From using Plemelj formula, one may find that ψ is in fact only expressed in terms of known data. More specifically, one may show that

$$\lim_{\lambda\to e^{i\theta}, \lambda\in D_\pm} \int_{\mathbb{R}^2} G_\pm(\mathrm{x} - \mathrm{y}, \lambda) f(\mathrm{y})\ dy = \pm\frac{1}{2i}(HR_\theta f)(\mathrm{x}\cdot\boldsymbol{\theta}^\perp) + (D_\theta f)(\mathrm{x}),$$

where $D_\theta f(\mathrm{x}) := \frac{1}{2}\int_{\mathbb{R}} \mathrm{sign}\ (t) f(\mathrm{x} + t\boldsymbol{\theta})\ dt$ is the so-called *divergent beam transform*. This term is of course unknown from data, but disappears when computing the jump of u across T.

3. To obtain a reconstruction formula for $f(z)$, we now set $\lambda \to 0$ in the "complex transport" equation to obtain

$$f(z) = \lim_{\lambda\to 0} \lambda^{-1}\partial_{\bar z} u(z, \lambda).$$

   Combining the expression of u in terms of ψ and of ψ in terms of $R_\theta f$, we recover the usual Filtered-Backprojection formula.

3.4.5 The attenuated X-ray transform

One of the powers of this method is that it can also deal with the presence of an attenuation coefficient. More specifically, suppose an attenuation function $a \in C_c^\infty(\mathbb{R}^2)$ is known, and for $f \in C_c^\infty(\mathbb{R}^2)$, let us consider the transform $I_a f(\mathrm{x}, \theta) = \lim_{t\to\infty} u(\mathrm{x} + t\boldsymbol{\theta}, \theta)$, where u solves the transport problem

$$\theta \cdot \nabla u + au = f \qquad (\mathbb{R}^2 \times \mathbb{S}^1), \qquad \lim_{t\to\infty} u(\mathrm{x} - t\boldsymbol{\theta}, \theta) = 0.$$

Upon complexifying $\lambda = e^{i\theta}$, the equation becomes

$$(\lambda\partial_z + \lambda^{-1}\partial_{\bar z})u(z,\lambda) + a(z)u(z,\lambda) = f(z).$$

Here it is then customary to introduce $h(z,\lambda)$, for $\lambda \notin T \cup \{0\}$, the unique solution to the problem

$$(\lambda\partial_z + \lambda^{-1}\partial_{\bar z})h = a, \qquad \lim_{|z|\to\infty} h(z,\lambda) = 0,$$

so that the complex transport equation becomes

$$(\lambda\partial_z + \lambda^{-1}\partial_{\bar z})(ue^h) = ae^h.$$

As in the unattenuated problem, $h_\pm$ are holomorphic in λ and the jump of $h_\pm$ is known (in fact, even the radial limits of $h_\pm$ are known since a is known).

Solving the last equation gives two expressions of u for $\lambda \in D_\pm$, that is,

$$u_\pm(z,\lambda) = e^{-h_\pm(z,\lambda)} \int_{\mathbb{C}} G_\pm(z - z', \lambda) e^{h_\pm(z',\lambda)} f(z')\ d\mu(z').$$

Again, it can then be shown that such an expression, with z frozen, is sectionally analytic in λ, with limit 0 at $\lambda = 0$, and with the right decay at infinity. Then u can be parameterized from its jump across T which can be expressed in terms of known data (that this step is true requires some careful calculations).

Finally, the reconstruction formula for f will be again obtained from computing the expression

$$f(z) = \lim_{\lambda\to 0} \lambda^{-1}\partial_{\bar z} u(z,\lambda).$$

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