Inverse boundary problems for semilinear elliptic PDE and linearized anisotropic Calderón problem

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Non-linearity helps in solving inverse problems!

- Hyperbolic case: Kurylev–Lassas–Uhlmann, 2018, Lassas–Uhlmann–Wang, 2019, Sá Barreto–Wang, 2018, Chen–Lassas–Oksanen–Paternain, 2019, 2020, Hintz–Uhlmann, 2018, Feizmohammadi–Lassas–Oksanen, 2020, Lassas–Liimatainen–Potenciano-Machado–Tyni, 2020, Hintz–Uhlmann–Zhai, 2020, ...
- Elliptic case: Feizmohammadi–Oksanen; Lassas–Liimatainen–Lin–Salo, 2019, K.–Uhlmann, 2019, Ma–Tzou, 2020, Lai–Zhou, 2020, Cârstea–Feizmohammadi, 2020, ...

A common feature of these works is that the presence of a nonlinearity allows one to solve inverse problems for non-linear equations in cases where the corresponding inverse problem in the linear setting is open.

The purpose of the first part of the talk is to demonstrate this phenomenon for inverse boundary problems for semilinear elliptic PDE in both full and partial data cases.

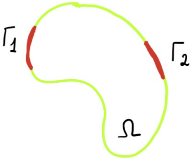
The Calderón problem with partial data

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with smooth boundary, and let $q \in C^\infty(\overline{\Omega})$. Consider the Dirichlet problem for the Schrödinger equation,

$$\begin{cases}(-\Delta + q)u = 0 & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = f. \end{cases}$$

Assume that 0 is not an eigenvalue of $-\Delta + q$. Let $\Gamma_1, \Gamma_2 \subset \partial\Omega$ be arbitrary open non-empty. The partial Dirichlet–to–Neumann map,

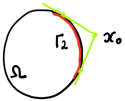
$$\Lambda_q^{\Gamma_1,\Gamma_2}(f) = \partial_\nu u|_{\Gamma_2}, \quad \text{supp } (f) \subset \Gamma_1.$$



The Calderón problem with partial data: Does $\Lambda_q^{\Gamma_1,\Gamma_2}$ determine q in Ω?

Known results

- $n \geq 3$: the problem is open in general. The problem is solved when
  - $\Gamma_2 = \{x \in \partial\Omega : \frac{(x-x_0)}{|x-x_0|} \cdot \nu(x) < \varepsilon\}$, $x_0 \notin \overline{ch(\Omega)}$, $\quad \varepsilon > 0$, $\Gamma_1 = \text{neigh}(\partial\Omega \setminus \Gamma_2, \partial\Omega)$ (Kenig–Sjöstrand–Uhlmann, 2007).



Note: when Ω is strictly convex, Γ_2 could be arbitrarily small

- $n = 2$: open when $\Gamma_1 \cap \Gamma_2 = \emptyset$. The problem is solved
  - when $\Gamma_1 = \Gamma_2$ is an arbitrary open non-empty portion of $\partial\Omega$ and $q \in C^{1,\alpha}(\overline{\Omega})$ (Imanuvilov–Uhlmann–Yamamoto, 2010, Guillarmou–Tzou, 2011 (in the case of Riemann surfaces)),
  - when $\Gamma_1 \cap \Gamma_2 = \emptyset$, provided that some additional geometric assumptions are satisfied, and $q \in C^{2,\alpha}(\overline{\Omega})$ (Imanuvilov–Uhlmann–Yamamoto, 2011).

Counterexamples to non-uniqueness for anisotropic Calderón problem in the case of measurements on disjoint sets: Daudé–Kamran–Nicoleau, 2019, 2020.

Partial data inverse boundary problems for semilinear elliptic PDE

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a connected bounded open set with C^∞ boundary.

Consider the Dirichlet problem for the following semilinear elliptic equation,

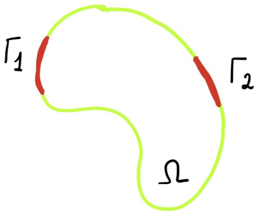
$$\begin{cases} -\Delta u + q(x)u^m = 0 \quad \text{in} \quad \Omega, \\ u = f \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $m \geq 2$, $q \in C^\alpha(\overline{\Omega})$, $0 < \alpha < 1$ (the Hölder space).

There exist $\delta > 0$ and $C > 0$ such that when $f \in B_\delta(\partial\Omega) := \{f \in C^{2,\alpha}(\partial\Omega) : \|f\|_{C^{2,\alpha}(\partial\Omega)} < \delta\}$, the problem (1) has a unique solution $u = u_f \in C^{2,\alpha}(\overline{\Omega})$ satisfying $\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\delta$.

Let $\Gamma_1, \Gamma_2 \subset \partial\Omega$ be arbitrary open non-empty. Define the partial Dirichlet–to–Neumann map,

$$\Lambda_q^{\Gamma_1,\Gamma_2}(f) = \partial_\nu u|_{\Gamma_2}, \quad \text{supp}\,(f) \subset \Gamma_1.$$



Theorem (K.–Uhlmann; Lassas–Liimatainen–Lin–Salo, 2019)
$\Lambda_{q_1}^{\Gamma_1,\Gamma_2} = \Lambda_{q_2}^{\Gamma_1,\Gamma_2} \implies q_1 = q_2$ *in* Ω.

Remark. We can also consider more general semilinear elliptic equations,

$$-\Delta u + V(x,u) = 0 \quad \text{in} \quad \Omega,$$

where the function $V : \overline{\Omega} \times \mathbb{C} \to \mathbb{C}$ satisfies the following conditions:

(i) the map $\mathbb{C} \ni z \mapsto V(\cdot, z)$ is holomorphic with values in $C^\alpha(\overline{\Omega})$, for some $0 < \alpha < 1$,

(ii) $V(x,0) = \partial_z V(x,0) = 0$, for all $x \in \overline{\Omega}$.

Hence,

$$V(x,z) = \sum_{k=2}^{\infty} V_k(x) \frac{z^k}{k!}, \quad V_k(x) := \partial_z^k V(x,0) \in C^\alpha(M),$$

which converges in the $C^\alpha(\overline{\Omega})$ topology.

Theorem (K.–Uhlmann; Lassas–Liimatainen–Lin–Salo, 2019)
$\Lambda_{V_1}^{\Gamma_1,\Gamma_2} = \Lambda_{V_2}^{\Gamma_1,\Gamma_2} \Longrightarrow V_1 = V_2$ *in* $\Omega \times \mathbb{C}$.

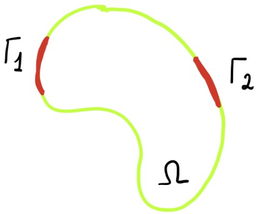
Consider next the following Dirichlet problem,

$$\begin{cases} -\Delta u + q(x)(\nabla u)^2 = 0 \quad \text{in} \quad \Omega, \\ u = f \quad \text{on} \quad \partial\Omega. \end{cases}$$

Here $q \in C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$, $(\nabla u)^2 = \nabla u \cdot \nabla u$.

For any $f \in C^{2,\alpha}(\partial\Omega)$ small, there exists a unique small solution $u \in C^{2,\alpha}(\overline{\Omega})$. Define the partial Dirichlet–to–Neumann map,

$$\Lambda_q^{\Gamma_1,\Gamma_2}(f) = \partial_\nu u|_{\Gamma_2}, \quad \text{supp } (f) \subset \Gamma_1.$$



Theorem (K.–Uhlmann, 2019)

$\Lambda_{q_1}^{\Gamma_1,\Gamma_2} = \Lambda_{q_2}^{\Gamma_1,\Gamma_2} \Longrightarrow q_1 = q_2$ *in* Ω.

Remark. Slightly more general nonlinearities can also be treated.

Idea of the proof (higher order linearization: Feizmohammadi–Oksanen; Lassas–Liimatainen–Lin–Salo, 2019)

Consider first, for $j = 1, 2$,

$$\begin{cases} -\Delta u_j + q_j(x) u_j^m = 0 & \text{in} \quad \Omega, \\ u_j = f & \text{on} \quad \partial\Omega, \end{cases}$$

Let $m = 2$ and let us perform a second order linearization of this problem.

To that end, let $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{C}^2$, and let $f_j \in C^\infty(\partial\Omega)$, supp $(f_j) \subset \Gamma_1$, $j = 1, 2$. The problem

$$\begin{cases} -\Delta u_j + q_j(x) u_j^2 = 0 & \text{in} \quad \Omega, \\ u_j = \varepsilon_1 f_1 + \varepsilon_2 f_2 & \text{on} \quad \partial\Omega, \end{cases}$$

has a unique small solution $u_j = u_j(\cdot, \varepsilon) \in C^{2,\alpha}(\overline{\Omega})$, which depends holomorphically on $\varepsilon \in \text{neigh}(0, \mathbb{C}^2)$ with values in $C^{2,\alpha}(\overline{\Omega})$.

Differentiating with respect to ε_l, $l = 1, 2$, taking $\varepsilon = 0$, and using that $u_j(x, 0) = 0$, we get the first order linearization,

$$\begin{cases} \Delta v_j^{(l)} = 0 \quad \text{in} \quad \Omega, \\ v_j^{(l)} = f_l \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $v_j^{(l)} = \partial_{\varepsilon_l} u_j|_{\varepsilon=0}$, $l = 1, 2$. By the uniqueness and the elliptic regularity for the Dirichlet problem, we see that $v^{(l)} := v_1^{(l)} = v_2^{(l)} \in C^\infty(\overline{\Omega})$, $l = 1, 2$.

Applying $\partial_{\varepsilon_1}\partial_{\varepsilon_2}|_{\varepsilon=0}$, we get the second order linearization,

$$\begin{cases} -\Delta w_j + 2q_j(x) v^{(1)} v^{(2)} = 0 \quad \text{in} \quad \Omega, \\ w_j = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

where $w_j = \partial_{\varepsilon_1}\partial_{\varepsilon_2} u_j|_{\varepsilon=0}$. The fact that

$$\Lambda_{q_1}^{\Gamma_1,\Gamma_2}(\varepsilon_1 f_1 + \varepsilon_2 f_2) = \Lambda_{q_2}^{\Gamma_1,\Gamma_2}(\varepsilon_1 f_1 + \varepsilon_2 f_2)$$

for all small $\varepsilon_1, \varepsilon_2$ and all $f_1, f_2 \in C^\infty(\partial\Omega)$ with supp (f_1), supp $(f_2) \subset \Gamma_1$ implies that $\partial_\nu w_1|_{\Gamma_2} = \partial_\nu w_2|_{\Gamma_2}$.

Multiplying the last equation by $v^{(3)} \in C^\infty(\overline{\Omega})$ harmonic in Ω and applying Green's formula, we get

$$2\int_\Omega (q_1 - q_2) v^{(1)} v^{(2)} v^{(3)} dx = \int_{\partial\Omega\setminus\Gamma_2} (\partial_\nu w_1 - \partial_\nu w_2) v^{(3)} dS = 0,$$

provided that supp $(v^{(3)}|_{\partial\Omega}) \subset \Gamma_2$. Hence, we obtain that

$$\int_\Omega (q_1 - q_2) v^{(1)} v^{(2)} v^{(3)} dx = 0,$$

for any $v^{(l)} \in C^\infty(\overline{\Omega})$ harmonic in Ω, $l = 1, 2, 3$, such that supp $(v^{(l)}|_{\partial\Omega}) \subset \Gamma_1$, $l = 1, 2$, and supp $(v^{(3)}|_{\partial\Omega}) \subset \Gamma_2$.
Take $v^{(3)} \not\equiv 0$.

Theorem (Dos Santos Ferreira–Kenig–Sjöstrand–Uhlmann, 2009)
Span $\{v^{(1)} v^{(2)} : v^{(l)} \in C^\infty(\overline{\Omega})$ *harmonic*, $v^{(l)}|_{\partial\Omega\setminus\Gamma_1} = 0, l = 1, 2\}$ *is dense in* $L^1(\Omega)$.

Using this result, we conclude that $q_1 = q_2$.

Let us now consider

$$\begin{cases} -\Delta u_j + q_j(x)(\nabla u_j)^2 = 0 & \text{in} \quad \Omega, \\ u_j = \varepsilon_1 f_1 + \varepsilon_2 f_2 & \text{on} \quad \partial\Omega. \end{cases}$$

Similarly, performing a second order linearization, we get

$$\int_\Omega (q_1 - q_2)(\nabla v^{(1)} \cdot \nabla v^{(2)}) v^{(3)} dx = 0,$$

for any $v^{(l)} \in C^\infty(\overline{\Omega})$ harmonic in Ω, $l = 1, 2, 3$, such that supp $(v^{(l)}|_{\partial\Omega}) \subset \Gamma_1$, $l = 1, 2$, and supp $(v^{(3)}|_{\partial\Omega}) \subset \Gamma_2$. Our inverse theorem follows therefore from the following density result.

Theorem (K.–Uhlmann, 2019)

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a connected bounded open set with C^∞ boundary, let $\Gamma \subset \partial\Omega$ be an open nonempty subset of $\partial\Omega$, and let $\tilde{\Gamma} = \partial\Omega \setminus \Gamma$. Then

$$Span\{\nabla u \cdot \nabla v : u, v \in C^\infty(\overline{\Omega}) \text{ harmonic}, u|_{\tilde{\Gamma}} = v|_{\tilde{\Gamma}} = 0\}$$

is dense in $L^1(\Omega)$.

Lai–Zhou, 2020: partial data inverse problem for nonlinear magnetic Schrödinger equation.

Inverse boundary problem for nonlinear magnetic Schrödinger equation

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary. To introduce the nonlinear magnetic Schrödinger operator, we consider 1-forms and scalar functions depending holomorphically on a parameter $z \in \mathbb{C}$. Specifically, let $A : M \times \mathbb{C} \to T^*M$ and $V : M \times \mathbb{C} \mapsto \mathbb{C}$ satisfy the conditions:

(A_i) the map $\mathbb{C} \ni z \mapsto A(\cdot, z)$ is holomorphic with values in $C^{1,1}(M, T^*M)$, the space of 1-forms with complex valued $C^{1,1}(M)$ coefficients,

(V_i) the map $\mathbb{C} \ni z \mapsto V(\cdot, z)$ is holomorphic with values in $C^{1,1}(M)$,

(V_{ii}) $V(x, 0) = 0$, for all $x \in M$.

We have

$$A(x,z) = \sum_{k=0}^{\infty} A_k(x) \frac{z^k}{k!}, \ A_k(x) := \partial_z^k A(x,0) \in C^{1,1}(M, T^*M),$$

$$V(x,z) = \sum_{k=1}^{\infty} V_k(x) \frac{z^k}{k!}, \ V_k(x) := \partial_z^k V(x,0) \in C^{1,1}(M).$$

Consider the nonlinear magnetic Schrödinger operator,

$$\begin{aligned} L_{A,V}u =& d^*_{A(\cdot,u)}d_{A(\cdot,u)}u + V(\cdot,u) \\ =& -\Delta_g u + d^*(iA(\cdot,u)u) - i\langle A(\cdot,u), du\rangle_g \\ & + \langle A(\cdot,u), A(\cdot,u)\rangle_g u + V(\cdot,u), \end{aligned}$$

for $u \in C^\infty(M)$. Here $d_A = d + iA$, $d : C^\infty(M) \to C^\infty(M, T^*M)$ is the de Rham differential, and d_A^* is the formal L^2–adjoint of d_A.

Notice that the first order linearization of $L_{A,V}$ is the standard linear magnetic Schrödinger operator $d^*_{A_0}d_{A_0} + V_1$.

Furthermore, we also assume that $A_0 \in C^\infty(M, T^*M)$, $V_1 \in C^\infty(M)$, and that 0 is not a Dirichlet eigenvalue of the operator $d^*_{A_0}d_{A_0} + V_1$.

Consider the Dirichlet problem for the nonlinear magnetic Schrödinger operator,

$$\begin{cases} L_{A,V} u = 0 \quad \text{in} \quad M, \\ u|_{\partial M} = f. \end{cases}$$

Under the above assumptions, there exist $\delta > 0$ and $C > 0$ such that when $f \in B_\delta(\partial M) := \{f \in C^{2,\alpha}(\partial M) : \|f\|_{C^{2,\alpha}(\partial M)} < \delta\}$, $0 < \alpha < 1$, the problem has a unique solution $u = u_f \in C^{2,\alpha}(M)$ satisfying $\|u\|_{C^{2,\alpha}(M)} < C\delta$. We define the Dirichlet–to–Neumann map

$$\Lambda_{A,V} f = \partial_\nu u_f|_{\partial M},$$

where $f \in B_\delta(\partial M)$ and ν is the unit outer normal to the boundary.

Inverse problem: Given $\Lambda_{A,V}$, determine the nonlinear magnetic and electric potentials, A and V, respectively.

When $A = 0$ and $V(x,z) = V_1(x)z$, the problem reduces to the inverse problem for the linear Schrödinger operator $-\Delta_g + V_1$ which asks: given Λ_{0,V_1}, determine V_1.

When $A = A_0(x)$ and $V(x,z) = V_1(x)z$, the problem reduces to the inverse problem for the linear magnetic Schrödinger operator $d^*_{\overline{A_0}} d_{A_0} + V_1$ which asks: given Λ_{A_0,V_1}, determine dA_0 and V_1.

- $n = 2$: the problem is solved (Imanuvilov–Uhlmann–Yamamoto, 2012, Guillarmou–Tzou, 2011).
- $n \geq 3$: the problem is open in general. Known results:
  - in the Euclidean setting (Sylvester–Uhlmann, 1987, Nakamura–Sun–Uhlmann, 1995),
  - for hyperbolic manifolds (Isozaki, 2004),
  - real analytic setting (Lee–Uhlmann, 1989, Kohn–Vogelius, 1984, Lassas–Uhlmann, 2001).

Going beyond these settings in dimension $n \geq 3$, the problem was only solved in the case when (M, g) is conformally transversally anisotropic (CTA) and under the assumption that the geodesic X-ray transform on the transversal manifold (M_0, g_0) is injective.

- in the case of Schrödinger operators: Dos Santos Ferreira–Kenig–Salo-Uhlmann, 2009, Dos Santos Ferreira–Kurylev–Lassas–Salo, 2016.
- in the case of magnetic Schrödinger operators: Cekić, 2017, K.–Uhlmann, 2018.

(M, g) is conformally transversally anisotropic (CTA) if there exists an $(n-1)$–dimensional smooth compact Riemannian manifold (M_0, g_0) with smooth boundary such that $M \subset\subset \mathbb{R} \times M_0$ and $g = c(e \oplus g_0)$ where e is the Euclidean metric on $\mathbb{R}$ and c is a positive smooth function on M.

Figure: Source: Mikko Salo's slides

Problem: remove the assumption that the geodesic X-ray transform on the transversal manifold (M_0, g_0) is injective.

- Open in the case of linear Schrödinger equation $-\Delta_g + V_1$.
- Known for $L_{0,V} u = -\Delta_g u + V(\cdot, u) = 0$.

Theorem (Feizmohammadi–Oksanen; Lassas–Liimatainen–Lin–Salo, 2019)

Let (M, g) be conformally transversally anisotropic manifold of dimension $n \geq 3$. If V satisfies the assumptions (V_i), (V_{ii}), and

(V_{iii}) $\partial_z V(x, 0) = \partial_z^2 V(x, 0) = 0$, *for all $x \in M$,*

then $\Lambda_{0,V}$ determines V in $M \times \mathbb{C}$ uniquely.

$$V(x, z) = \sum_{k=3}^{\infty} V_k(x) \frac{z^k}{k!}.$$

No assumptions on the transversal manifold.

The proof of this result relies on higher order linearizations of the Dirichlet–to–Neumann map, which allows one to reduce the inverse problem to the following density result,

Proposition (Lassas–Liimatainen–Lin–Salo, 2019)

Let (M, g) be a conformally transversally anisotropic manifold of dimension $n \geq 3$, and let $q \in C^{1,1}(M)$. If

$$\int_M q u_1 u_2 u_3 u_4 dV_g = 0,$$

for all harmonic functions $u_j \in C^\infty(M)$, $j = 1, 2, 3, 4$, then $q \equiv 0$.

We extend the result of Feizmohammadi–Oksanen; Lassas–Liimatainen–Lin–Salo, 2019 to the nonlinear magnetic Schrödinger equation,

$$\begin{aligned} L_{A,V}u =& d^*_{A(\cdot,u)} d_{A(\cdot,u)}u + V(\cdot,u) \\ =& -\Delta_g u + d^*(iA(\cdot,u)u) - i\langle A(\cdot,u), du\rangle_g \\ &+ \langle A(\cdot,u), A(\cdot,u)\rangle_g u + V(\cdot,u) = 0, \end{aligned}$$

Similarly to the assumption (V_{iii}) on V, we suppose that

(A_{ii}) $A(x,0) = \partial_z A(x,0) = 0$, for all $x \in M$.

Theorem (K.–Uhlmann, 2020)

Let (M,g) be a conformally transversally anisotropic manifold of dimension $n \geq 3$. Let $A^{(1)}, A^{(2)} : M \times \mathbb{C} \to T^*M$ and $V^{(1)}, V^{(2)} : M \times \mathbb{C} \mapsto \mathbb{C}$ satisfy the assumptions (A_i), (A_{ii}) and (V_i), (V_{ii}), (V_{iii}), respectively. If $\Lambda_{A^{(1)},V^{(1)}} = \Lambda_{A^{(2)},V^{(2)}}$ then $A^{(1)} = A^{(2)}$ and $V^{(1)} = V^{(2)}$ in $M \times \mathbb{C}$.

$$A(x,z) = \sum_{k=2}^{\infty} A_k(x)\frac{z^k}{k!}, \quad V(x,z) = \sum_{k=3}^{\infty} V_k(x)\frac{z^k}{k!}.$$

No assumptions on the transversal manifold whereas the corresponding inverse boundary problem for the linear magnetic Schrödinger operator is still open in this generality.

Idea of the proof

Similarly to Feizmohammadi–Oksanen; Lassas–Liimatainen–Lin–Salo, 2019, our proof relies on higher order linearizations of $\Lambda_{A,V}$, as well as a suitable consequence of the following density result, which may be of some independent interest.

Proposition (K.–Uhlmann, 2020)

Let (M, g) be a conformally transversally anisotropic manifold of dimension $n \geq 3$, and let $A \in C^{1,1}(M, T^*M)$ be a 1-form. If

$$\int_M \langle A, d(u_1 u_2 u_3)\rangle_g u_4 dV_g = 0, \tag{2}$$

for all harmonic functions $u_j \in C^\infty(M)$, $j = 1, 2, 3, 4$, then $A \equiv 0$.

Idea of the proof of Proposition in a simplified setting

We follow the strategy of Lassas–Liimatainen–Lin–Salo, 2019.

Assume that each point $p \in M_0^{\text{int}}$ is the unique intersection point of two distinct nontangential non-self-intersecting geodesics γ and η. Assume furthermore that the conformal factor $c = 1$.
Step 1. Boundary determination. Substitute $u_2 = u_3 = 1$ in the orthogonality relation (2).

Proposition
Let $A \in C^{1,1}(M, T^*M)$ be a 1-form. If

$$\int_M \langle A, du_1 \rangle_g u_2 dV_g = 0,$$

for all harmonic functions $u_1, u_2 \in C^\infty(M)$, then $A|_{\partial M} = 0$ and $\partial_\nu A|_{\partial M} = 0$.

This allows us to extend A by zero to $\mathbb{R} \times M_0 \setminus M$, while preserving its regularity.

By density, (2) also holds for all harmonic functions $u_j \in C^{2,\alpha}(M)$, $0 < \alpha < 1$, $j = 1, \dots, 4$.

Step 2. Let $s = \frac{1}{h}$, with $\lambda \in \mathbb{R}$ being fixed. We construct harmonic functions $u_j \in C^3(M)$, $j = 1, \dots, 4$, on (M, g) of the form

$$u_1 = e^{-(s+i\lambda)x_1}(v + r_1), \quad u_2 = \overline{e^{(s+i\lambda)x_1}(v + r_2)},$$
$$u_3 = e^{-sx_1}(w + r_3), \quad u_4 = \overline{e^{sx_1}(w + r_4)},$$

where

$$\|r_j\|_{C^1(M)} = \mathcal{O}(s^{-K}),$$

as $s \to \infty$, $K \gg 1$. Here $v = v(\cdot; s), w = w(\cdot; s) \in C^\infty(M_0)$ are Gaussian beams quasimodes concentrating near the geodesics η and γ, respectively.

$$\|v\|_{L^4(M_0)} = \|w\|_{L^4(M_0)} = \mathcal{O}(1),$$

as $s \to \infty$.

We have

$$v(x';s)=s^{\frac{n-2}{8}}e^{i(s+i\lambda)\varphi(x')}a(x';s),\quad w(x';s)=s^{\frac{n-2}{8}}e^{is\psi(x')}b(x';s),$$

where

$$\varphi(\eta(t))=t,\ \nabla\varphi(\eta(t))=\dot\eta(t),\ \mathrm{Im}\,(\nabla^2\varphi(\eta(t)))\geq 0,\ \mathrm{Im}\,(\nabla^2\varphi)|_{\dot\eta(t)^\perp}>0,$$
$$\psi(\gamma(\tau))=\tau,\ \nabla\psi(\gamma(\tau))=\dot\gamma(\tau),\ \mathrm{Im}\,(\nabla^2\psi(\gamma(\tau)))\geq 0,\ \mathrm{Im}\,(\nabla^2\psi)|_{\dot\gamma(\tau)^\perp}>0,$$

and

$$a(t,y;s)=\left(\sum_{j=0}^{N}\tau^{-j}a_j\right)\chi\left(\frac{y}{\delta'}\right),\quad b(\tau,z;s)=\left(\sum_{j=0}^{N}\tau^{-j}b_j\right)\chi\left(\frac{z}{\delta'}\right),$$

$$a_0(t,y)=a_{00}^{(l)}(t)+\mathcal{O}(|y|),\quad a_{00}(t)\neq 0,\quad \forall t,$$
$$b_0(\tau,z)=a_{00}(\tau)+\mathcal{O}(|z|),\quad b_{00}(\tau)\neq 0,\quad \forall\tau.$$

Here (t,y) and (τ,z) are the Fermi coordinates for the geodesics η and γ, $\chi\in C_0^\infty(\mathbb{R}^{n-2})$ is such that $0\leq\chi\leq 1$, $\chi=1$ for $|y|\leq 1/4$ and $\chi=0$ for $|y|\geq 1/2$, and $\delta'>0$ is a fixed number.

Step 3. Writing $A = (A_1, A')$, substituting our harmonic functions into (2), and denoting the partial Fourier transform of A in the x_1 variable by $\hat{A}(\lambda, x')$, we get

$$\int_{M_0} (-s\hat{A}_1(2\lambda, \cdot)|v|^2|w|^2 + \langle \hat{A}'(2\lambda, \cdot), d_{x'}(|v|^2 w)\overline{w}\rangle_{g_0}) dV_{g_0} = \mathcal{O}(1),$$

as $s \to \infty$. Since v and w can be chosen to be supported in arbitrarily small but fixed neighborhoods of η and γ, respectively, and since η and γ only intersect at p, the products $|v|^2|w|^2$ and $d_{x'}(|v|^2 w)\overline{w}$ concentrate in a small neighborhood U of p.

Using the expressions for the Gaussian beams v and w, and dividing by $s^{\frac{1}{2}}$, we obtain that

$$s^{\frac{n-1}{2}} \int_U (-\hat{A}_1(2\lambda, \cdot) + i\langle \hat{A}'(2\lambda, \cdot), 2id \operatorname{Im}\varphi + d\psi\rangle_{g_0}) e^{-2\lambda \operatorname{Re}\varphi} |a_0|^2 |b_0|^2 e^{-s\Psi} dV_{g_0} = \mathcal{O}(s^{-\frac{1}{2}}),$$

as $s \to \infty$, where

$$\Psi = 2(\operatorname{Im}\varphi + \operatorname{Im}\psi) \geq 0.$$

By the properties of the phases of our Gaussian beams, we have

$$\Psi(p) = 0, \quad d\Psi(p) = 0, \quad \nabla^2\Psi(p) > 0,$$

where the later inequality is a consequence of the fact that the Hessians of $\operatorname{Im}\varphi$ and $\operatorname{Im}\psi$ at p are positive definite in the directions orthogonal to η and γ, respectively.
Passing to the limit as $s \to \infty$ and using the rough version of stationary phase lemma (Laplace method), we get

$$(-\hat{A}_1(2\lambda, p) + i\hat{A}'(2\lambda, p)(\dot{\gamma}(t_0))e^{-2\lambda \operatorname{Re}\varphi(p)}|a_{00}(p)|^2|b_{00}(p)|^2 = 0,$$

where $p = \gamma(t_0)$, for all $\lambda \in \mathbb{R}$. As $a_{00}(p) \neq 0$, $b_{00}(p) \neq 0$, and λ is arbitrary, we see that

$$-A_1(x_1, p) + iA'(x_1, p)(\dot{\gamma}(t_0)) = 0,$$

which is equivalent to

$$(iA_1, A')(x_1, p)(1, \dot{\gamma}(t_0)) = 0.$$

Spanning the tangent space T_pM_0 by the tangent vectors of the geodesics which are small perturbations of γ, we get $A = 0$. The proof of Proposition in the simplified case is complete.

In the case of a general transversal manifold M_0, the non-tangential geodesics γ and η might have self-intersections and may intersect in more than one point, which complicates the proof. Here additionally to the stationary phase argument, we also rely on non-stationary phase, where we have to integrate by parts twice. This is precisely the motivation for our regularity assumption on $A \in C^{1,1}(M, T^*M)$.

Back to nonlinear magnetic Schrödinger equation

Considering the mth order linearization, $m \geq 3$, leads to the following integral identity,

$$\int_M \big((m+1)i\langle A, d(u_1 \cdots u_m)\rangle_g u_{m+1} dV_g = \big(mid^*(A) + V\big)u_1 \cdots u_{m+1}\big) dV_g,$$

where $A = A^{(1)}_{m-1} - A^{(2)}_{m-1}$ and $V = V^{(1)}_m - V^{(2)}_m$, which is valid for any $u_j \in C^{2,\alpha}(M)$ harmonic, $j = 1, \ldots, m+1$. Setting $u_1 = \cdots = u_{m-3} = 1$ gives the identity

$$(m+1)i\int_M \langle A, d(u_{m-2}u_{m-1}u_m)\rangle_g u_{m+1} dV_g = \int_M \big(mid^*(A) + V\big) u_{m-2}u_{m-1}u_m u_{m+1}\big) dV_g.$$

We first show that $A|_{\partial M} = 0$ and $\partial_\nu A|_{\partial M} = 0$, and then use a consequence of our Proposition to obtain that $A \equiv 0$. To recover V, we let $A = 0$, and rely on the density result of Lassas–Liimatainen–Lin–Salo, 2019.

Back to the Calderón problem for $-\Delta_g + V_1$

The Calderón problem is solved under the assumption that (M, g) is CTA of dimension $n \geq 3$ and the geodesic X-ray transform on (M_0, g_0) is injective. (Dos Santos Ferreira–Kenig–Salo–Uhlmann, 2009, Dos Santos Ferreira–Kurylev–Lassas–Salo, 2016).

Problem. Remove or propose an alternative condition to the injectivity of the geodesic X-ray transform.

This can be done for the linearized version (at $V_1 = 0$) of the above problem:
If $f \in L^\infty(M)$ satisfies

$$\int_M f u_1 u_2 \, dV_g = 0$$

for all $u_j \in L^2(M)$ with $-\Delta_g u_j = 0$ in M, $j = 1, 2$, is it true that $f \equiv 0$?

Geometric condition on transversal manifold

Definition (Dos Santos Ferreira–Kurylev–Lassas–Liimatainen–Salo, 2017)

We say that $(x_0', \xi_0') \in S^* M_0^{\text{int}}$ is generated by an admissible pair of geodesics, if there are two nontangential unit speed geodesics

$$\gamma_1 : [-T_1, T_2] \to M_0, \quad \gamma_2 : [-S_1, S_2] \to M_0,$$

$0 < T_1, T_2, S_1, S_2 < \infty$, such that

(i) $\gamma_1(0) = \gamma_2(0) = x_0'$,

(ii) $\dot\gamma_1(0) + \dot\gamma_2(0) = t_0 \xi_0'$, for some $0 < t_0 < 2$, where ξ_0' is understood as an element of $T_{x_0} M_0^{\text{int}}$ by the Riemannian duality,

(iii) γ_1, γ_2 do not have self-intersections at the point x_0', and x_0' is the only point of their intersections, i.e.

$$\gamma_1(t) = x_0' \Leftrightarrow t = 0, \quad \gamma_2(s) = x_0' \Leftrightarrow s = 0,$$
$$\gamma_1(t) = \gamma_2(s) \Rightarrow \gamma_1(t) = \gamma_2(s) = x_0'.$$

We have the following analytic microlocal result, which is an analog of the result of Dos Santos Ferreira–Kurylev–Lassas–Liimatainen–Salo, 2017, established in the C^∞–case.

Theorem (K.–Liimatainen–Salo, 2020)

Let (M,g) be a transversally anisotropic manifold and assume that the transversal manifold (M_0,g_0) is real analytic with real analytic boundary and real analytic metric g_0, up to the boundary. Assume furthermore that $f\in L^\infty(M)$ satisfies

$$\int_M f u_1 u_2 \, dV_g = 0,$$

for all $u_j\in L^2(M)$ with $-\Delta_g u_j=0$ in M^{int}. Let $(x_0',\xi_0')\in S^*M_0^{int}$ be generated by an admissible pair of geodesics. Then for any $\lambda\in\mathbb{R}$, one has

$$(x_0',\xi_0')\notin WF_a(\hat{f}(\lambda,\cdot))\subset T^*M_0^{int}\setminus\{0\}.$$

Here we extend $f\in L^\infty(M)$ by zero to $(\mathbb{R}\times M_0)\setminus M$. Writing $x=(x_1,x')$ where $x_1\in\mathbb{R}$, and x' are local coordinates M_0, we let

$$\hat{f}(\lambda,x')=\int_{-\infty}^{\infty} e^{-i\lambda x_1} f(x_1,x')\,dx_1,\quad \lambda\in\mathbb{R},$$

be the Fourier transform of f with respect to x_1.

We have the following global result, which gives the positive answer to the linearized Calderón problem under suitable geometric assumptions.

Theorem (K.–Liimatainen–Salo, 2020)

Let (M, g) be a transversally anisotropic manifold and assume that the transversal manifold (M_0, g_0) is connected, M_0^{int} as well as g_0 in M_0^{int} are real analytic. Assume that every point $(x_0', \xi_0') \in S^*M_0^{int}$ is generated by an admissible pair of geodesics. Moreover, assume that $f \in L^\infty(M)$ satisfies

$$\int_M f u_1 u_2 \, dV_g = 0,$$

for all $u_j \in L^2(M)$ with $-\Delta_g u_j = 0$ in M^{int}. Then $f = 0$ in M.

Note that while (M_0, g_0) is real analytic, our Theorem does not follow from the existing results in the real analytic setting, as it corresponds to deforming the zero potential by an L^∞ perturbation.

There are transversally anisotropic manifolds (M, g) with a transversal manifold (M_0, g_0) satisfying the geometric conditions of our Theorem and with a non-invertible geodesic X-ray transform. Therefore, the geometric Calderón problem is still open on such manifolds while our Theorem gives a positive solution to the corresponding linearized problem.

Example

Let $M_0 = \mathbb{S}^1 \times [0, a]$, $a > 0$, be a cylinder with its usual flat metric g_0. The geodesics on M_0 are straight lines, circular cross sections, and helices that wind around the cylinder. The geodesic X-ray transform is not invertible, since the kernel contains functions of the form $f(e^{it}, s) = h(s)$ where $h \in C_0^\infty((0, a))$ integrates to zero over $[0, a]$. However, one can check that every point $(x_0', \xi_0') \in S^*M_0^{\text{int}}$ is generated by an admissible pair of geodesics.

Gaussian beam quasimodes with exponentially small errors

We have the following essentially well known result (see Sjöstrand, 1975, 1982, Babich, 1996).

Theorem

Let (X, g) be a compact real analytic Riemannian manifold of dimension $n \geq 2$ with real analytic boundary and real analytic metric g, up to the boundary. Let $\gamma : [-T_1, T_2] \to X$, $0 < T_1, T_2 < \infty$, be a unit speed non-tangential geodesic, and let $\lambda \in \mathbb{R}$. There is a family of C^∞ functions $v(x; h)$ on X, $0 < h \leq 1$, and $C > 0$ such that supp $(v(\cdot; h))$ is confined to a small neighborhood of $\gamma([-T_1, T_2])$ and

$$\|(-h^2\Delta_g - (hs)^2)v\|_{L^2(X)} = \mathcal{O}(e^{-\frac{1}{Ch}}), \quad \|v\|_{L^2(X)} \asymp 1,$$

as $h \to 0$. Here $s = \frac{1}{h} + i\lambda$.

Theorem (continuation)

The local structure of the family $v(x;h)$ is as follows: let $p \in \gamma([-T_1, T_2])$ and let $t_1 < \cdots < t_{N_p}$ be the times in $(-T_1, T_2)$ when $\gamma(t_l) = p$, $l = 1, \ldots, N_p$. In a sufficiently small neighborhood V of a point $p \in \gamma([-T_1, T_2])$, we have

$$v|_V = v^{(1)} + \cdots + v^{(N_p)},$$

where each $v^{(l)}$ has the form

$$v^{(l)}(x;h) = h^{-\frac{(n-1)}{4}} e^{is\varphi^{(l)}(x)} a^{(l)}(x;h).$$

Here $\varphi = \varphi^{(l)}$ is real analytic in V satisfying for t near t_l,

$$\varphi(\gamma(t)) = t,\ \nabla\varphi(\gamma(t)) = \dot{\gamma}(t),\ Im\,(\nabla^2\varphi(\gamma(t))) \geq 0,\ Im\,(\nabla^2\varphi)|_{\dot{\gamma}(t)^\perp} > 0,$$

and $a^{(l)}$ is an elliptic classical analytic symbol in a complex neighborhood of p.

Idea of the construction of quasimodes

Boutet de Monvel–Krée, 1967, Sjöstrand, 1982: Let $V \subset \mathbb{C}^n$ be an open set. We say that $a(x;h) = \sum_{k=0}^{\infty} h^k a_k(x)$ is a (formal) classical analytic symbol in V if $a_k \in \mathrm{Hol}(V)$, $k = 0, 1, 2, \ldots$, and for every $\tilde{V} \subset\subset V$, there exists $C = C_{\tilde{V}} > 0$ such that

$$|a_k(x)| \leq C^{k+1} k^k, \quad x \in \tilde{V},$$

$k = 0, 1, 2, \ldots$. The classical analytic symbol $a(x;h)$ is said to be elliptic if $a_0 \neq 0$.

We work in the Fermi coordinates where the γ is given by $\{x = (t, y) : y = 0\}$. Consider the following Gaussian beam ansatz,

$$v(t, y; h) = e^{is\varphi(t,y)} a(t, y; h), \quad s = \frac{1}{h} + i\lambda, \quad \lambda \in \mathbb{R},$$

where the phase φ is complex valued with $\operatorname{Im} \varphi(t, y) \geq 0$ and a is an amplitude. As usual, this leads to solving the eikonal equation for φ and a sequence of transport equations for a.

In the usual Gaussian beam construction in the C^{∞}–setting, one solves the eikonal equation to a large, and sometimes infinite, order along the geodesic. Working in the real analytic setting, it is natural to solve the eikonal equation,

$$\langle d\varphi, d\varphi\rangle_g - 1 = p(x, \varphi_x'(x)) = 0.$$

in a complex neighborhood of a geodesic segment of γ. Here $p(x,\xi) = |\xi|^2_{g(x)} - 1$, holomorphically continued to the complex domain. When solving the Hamilton-Jacobi equation, we proceed by a geometric argument, constructing a suitable complex Lagrangian manifold $\Lambda \subset p^{-1}(0)$ in a complex neighborhood of the null bicharacteristic corresponding to the geodesic segment. Furthermore, we construct Λ so that it is positive and so that it stays transversal to the fiber along the entire bicharacteristic. The solution φ is then obtained as a generating function of Λ.

We next show, by adapting the nested neighborhood method of Sjöstrand, 1982, to cover the entire geodesic segment, that the solution $a(x, h)$ of the transport equations is given by an elliptic classical analytic symbol. Thus, we get for $N = 1, 2, \dots,$

$$e^{-is\varphi}(-h^2\Delta_g - (hs)^2)e^{is\varphi}\left(\sum_{j=0}^{N} h^j a_j\right) = h^{N+2}T_2(a_N), \quad (3)$$

where T_2 is a second order differential operator with analytic coefficients. Cauchy's estimates and the definition of analytic symbol then yield that the right hand side in (3) is bounded by $h^{N+2}C^{N+1}N^N$. Letting the order N of the expansions of a depend on h as $N = N(h) = [\frac{1}{heC}]$ gives the exponentially small error in the theorem.

Remark. Here our quasimodes construction is performed along the entire geodesic segment contrary to the standard constructions in a neighborhood of a point by Dencker–Sjöstrand–Zworski, 2004.

Idea of the proof of analytic microlocal result

Let $\alpha_0 = (x_0', \xi_0') \in S^*M_0^{int}$ be generated by an admissible pair of geodesics $\gamma_1(\alpha_0)$ and $\gamma_2(\alpha_0)$ on M_0.
Step 1. Show that there exists a neighborhood V of α_0 in $S^*M_0^{int}$ such that every point $\alpha = (\alpha_{x'}, \alpha_{\xi'}) \in V$ is generated by an admissible pair of geodesics $\gamma_1(\alpha)$ and $\gamma_2(\alpha)$ on M, which depend real-analytically on α.

Step 2. Let $s_1 = \frac{1}{h} + i\lambda$ and $s_2 = \frac{1}{h}$, where $\lambda \in \mathbb{R}$. We construct two families of Gaussian beam quasimodes $v_1(x', \alpha; h)$ and $v_2(x', \alpha; h)$, associated to $\gamma_1(\alpha)$ and $\gamma_2(\alpha)$, depending real analytically on $\alpha \in V$ such that

$$\|v_j(\cdot, \alpha; h)\|_{L^2(M)} \asymp 1, \ \|(-h^2\Delta_{g_0} - (hs)^2)v_j(\cdot, \alpha; h)\|_{L^2(M)} = \mathcal{O}(e^{-\frac{1}{Ch}}),$$

as $h \to 0$, for some $C > 0$, uniformly in $\alpha \in V$.

The fact that (M, g) is transversally anisotropic provides us with the limiting Carleman weight $\phi(x) = x_1$ for the Laplacian, and using the technique of Carleman estimates, we convert the families of Gaussian beams v_1 and v_2 into two families of harmonic functions on M,

$$u_1(x, \alpha; h) = e^{-s_1 x_1}(v_1(x', \alpha; h) + r_1(x, \alpha; h)),$$
$$u_2(x, \alpha; h) = e^{s_2 x_1}(v_2(x', \alpha; h) + r_2(x, \alpha; h)),$$

where

$$\|r_j\|_{L^2(M)} = \mathcal{O}(e^{-\frac{1}{Ch}}), \quad C > 0,$$

as $h \to 0$, uniformly in $\alpha \in V$.

Step 3. Testing the orthogonally relation with the constructed families of harmonic functions leads to

$$\int_M f e^{-i\lambda x_1} v_1(x', \alpha; h) v_2(x', \alpha; h) dV_g = \mathcal{O}(e^{-\frac{1}{Ch}}), \quad C > 0,$$

uniformly in $\alpha \in V$. Extending $f \in L^\infty(M)$ by zero to $(\mathbb{R} \times M_0) \setminus M$, we obtain that

$$\int_{M_0} \hat{f}(\lambda, x') v_1(x', \alpha; h) v_2(x', \alpha; h) dV_{g_0} = \mathcal{O}(e^{-\frac{1}{Ch}}), \quad C > 0,$$

uniformly in $\alpha \in V$.

Recalling that the geodesics $\gamma_1(\alpha)$ and $\gamma_2(\alpha)$ intersect at $\alpha_{x'}$ only and that

$$\text{supp } (v_j(\cdot, \alpha; h)) \subset \text{small neigh}(\gamma_j(\alpha)), \quad j = 1, 2,$$

we get

$$\int_{\text{neigh}(\alpha_{x'}, M_0)} \hat{f}(\lambda, x') v_1(x', \alpha; h) v_2(x', \alpha; h) \sqrt{g_0(x')} dx' = \mathcal{O}(e^{-\frac{1}{Ch}}),$$

uniformly in $\alpha \in V$. Recalling that the geodesics $\gamma_1(\alpha)$, $\gamma_2(\alpha)$ do not have self-intersections at $\alpha_{x'}$, we have in a small neighborhood of $\alpha_{x'}$,

$$v_1(x', \alpha; h) = h^{-\frac{(n-2)}{4}} e^{is_1 \varphi_1(x', \alpha)} a_1(x', \alpha; h),$$
$$v_2(x', \alpha; h) = h^{-\frac{(n-2)}{4}} e^{is_2 \varphi_2(x', \alpha)} a_2(x', \alpha; h).$$

Thus, we have

$$\int_{\text{neigh}(\alpha_{x'},M_0)} e^{\frac{i\varphi(x',\alpha)}{h}} \hat{f}(\lambda,x')a(x',\alpha;h)dx' = \mathcal{O}(e^{-\frac{1}{Ch}}), \quad h\to 0,$$

uniformly in $\alpha\in V$. Here $\varphi(x',\alpha)=\varphi_1(x',\alpha)+\varphi_2(x',\alpha)$ is analytic in a neighborhood of (x_0',α_0), and

$$a(x',\alpha;h)=e^{-\lambda\varphi_1(x',\alpha)}a_1(x',\alpha;h)a_2(x',\alpha;h)\sqrt{g_0(x')}$$

is an elliptic classical analytic symbol in a neighborhood of (x_0',α_0). We have

$$\varphi(x',\alpha)|_{x'=\alpha_{x'}}=0, \quad \varphi'_{x'}(x',\alpha)|_{x'=\alpha_{x'}}=t_0\alpha_{\xi'},$$
$$\operatorname{Im}\varphi(x,\alpha)\geq C_0|x-\alpha_x|^2, \quad x,\alpha \text{ real},$$

for some $C_0>0$.

Using the FBI characterization of the analytic wave front set of Sjöstrand, 1982, we conclude that $\alpha_0\notin \mathrm{WF}_a(\hat{f}(\lambda,\cdot))$ for all $\lambda\in\mathbb{R}$.

Note that we need to work with families of Gaussian beams to fill up an entire neighborhood of α_0.

THANK YOU VERY MUCH FOR YOUR ATTENTION!