Stability and Uniqueness for Some Nonlocal Calderón Type Inverse Problems

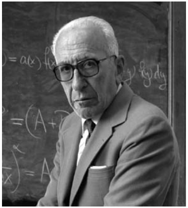
Angkana Rüland
(joint work with G. Covi, M.-Á. García-Ferrero, T. Ghosh, M. Salo, G. Uhlmann)

STRUCTURES
CLUSTER OF
EXCELLENCE

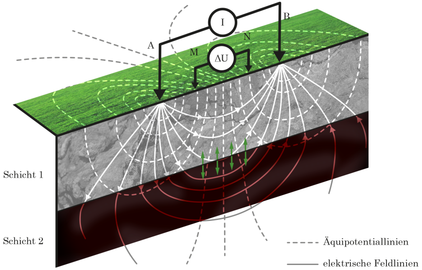
UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386

Inverse problems and nonlinearity, 25.08.2020

The Classical Calderón Problem



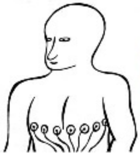
[picture from Wikipedia]



Q: Find location of oil by carrying out current-to-voltage measurements!

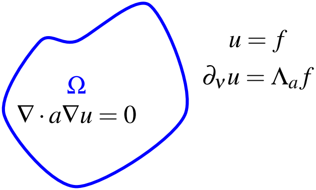
Applications e.g. in

- geoprospecting,
- medicine,
- detection of corrosion...



[picture from http://siltanen-research.net/project_EIT.html]

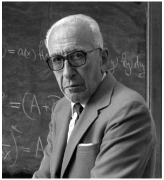
The Classical Calderón Problem



$$\nabla \cdot a\nabla u = 0 \text{ in } \Omega,$$
$$u = f \text{ on } \partial\Omega.$$

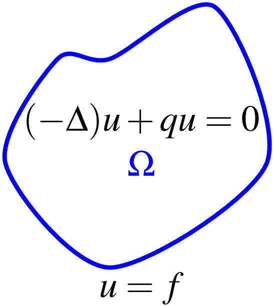
$$\Lambda_a : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega),$$
$$f \mapsto a\nabla u \cdot \nu.$$

- Ω conducting medium with conductivity a,
- f voltage at the boundary of Ω,
- u potential,
- $i(x) = -a(x)\nabla u(x)$ current (Ohm's law),
- $a\nabla u \cdot \nu|_{\partial\Omega}$ current flowing out of boundary.



[picture from Wikipedia]

The Classical Calderón Problem



Consider

$$(-\Delta)u + qu = 0 \text{ in } \Omega, \quad u = f \text{ on } \partial\Omega.$$

$$\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \quad f \mapsto \partial_\nu u.$$

Q: Possible to recover q from Dirichlet-to-Neumann map?

- Injectivity: $\Lambda_{q_1} = \Lambda_{q_2} \Rightarrow q_1 = q_2$ [Sylvester-Uhlmann, '87].
- (Conditional) Stability: $\Lambda_{q_1} \sim \Lambda_{q_2} \Rightarrow q_1 \sim q_2$ [Alessandrini, '88].
- Recovery: There exists a constructive scheme for reconstructing the potential q [Nachman, '88].

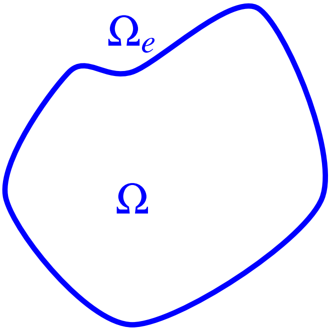
The Fractional Calderón Problem

Consider

$$(-\Delta)^s u + qu = 0 \text{ in } \Omega,$$
$$u = f \text{ in } \Omega_e,$$

where $\Omega_e = \mathbb{R}^n \setminus \Omega$, $s \in (0,1)$.

$$\Lambda_q(f) = (-\Delta)^s u|_{\Omega_e}$$



Q: Possible to recover q from Dirichlet-to-Neumann map? [Ghosh-Salo-Uhlmann '16]

Relevance:

- Intrinsic interest: effect of nonlocality [Caffarelli-Silvestre '07].
- Applications: water waves, crystal defects, biology, finance.
- Degenerate Robin inverse problem; construction of CGOs [Covi-R. '20].

Uniqueness

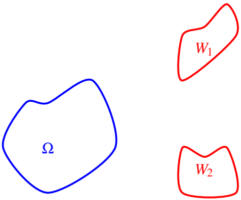
Theorem (Ghosh-Salo-Uhlmann, '16; R.-Salo '17)

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded open set. Let $s \in (0,1)$ and let $q_1, q_2 \in Z^{-s}(\Omega)$. Let $W_1, W_2 \subset \Omega_e$ be open subsets of Ω_e. If

$$\Lambda_{q_1} f|_{W_2} = \Lambda_{q_2} f|_{W_2} \text{ for any } f \in C_c^\infty(W_1),$$

then $q_1 = q_2$.

- **Partial data** result.
- Sharp **scaling critical** spaces.
- [R.-Salo '17; '18]: (Partial data) **stability estimates** with optimal logarithmic modulus.



Uniqueness Ideas

Theorem (Ghosh-Salo-Uhlmann '16, R.-Salo '17)

Let $\Omega \subset \mathbb{R}^n$*,* $n \geq 1$*, be a bounded open set. Let* $s \in (0,1)$ *and let* $q_1, q_2 \in Z^{-s}(\Omega)$*. Let* $W_1, W_2 \subset \Omega_e$ *be open subsets of* Ω_e*. If*

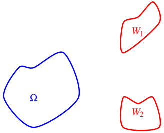
$$\Lambda_{q_1} f|_{W_2} = \Lambda_{q_2} f|_{W_2} \text{ for any } f \in C_c^\infty(W_1),$$

then $q_1 = q_2$*.*

- Alessandrini's identity:

$$((\Lambda_{q_1} - \Lambda_{q_2})f_1, f_2)_{W_2} = ((q_1 - q_2)u_1, u_2)_{\Omega}.$$

- Runge approximation ("All functions are locally s-harmonic up to a small error") [Dipiero-Savin-Valdinoci, Ghosh-Salo-Uhlmann, R.-Salo]; [Covi-R. '20].
- Robust; many extensions. [Ghosh-Lin-Xiao, Covi, Lai-Lin, Lai-Lin-R., Bhattacharya-Ghosh-Uhlmann, Cekić-Lin-R., Li, Covi-Mönkkönen-Railo-Uhlmann].



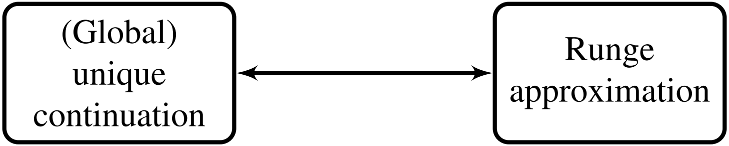
Uniqueness Ideas

Theorem (Ghosh-Salo-Uhlmann '16, R.-Salo '17)

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, *be a bounded open set. Let* $s \in (0,1)$ *and let* $q_1, q_2 \in Z^{-s}(\Omega)$. *Let* $W_1, W_2 \subset \Omega_e$ *be open subsets of* Ω_e. *If*

$$\Lambda_{q_1} f|_{W_2} = \Lambda_{q_2} f|_{W_2} \text{ for any } f \in C_c^{\infty}(W_1),$$

then $q_1 = q_2$.



Uniqueness Ideas

Theorem (Ghosh-Salo-Uhlmann '16, R.-Salo '17)

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded open set. Let $s \in (0,1)$ and let $q_1, q_2 \in Z^{-s}(\Omega)$. Let $W_1, W_2 \subset \Omega_e$ be open subsets of Ω_e. If

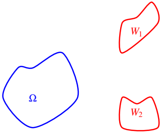
$$\Lambda_{q_1} f|_{W_2} = \Lambda_{q_2} f|_{W_2} \text{ for any } f \in C_c^\infty(W_1),$$

then $q_1 = q_2$.

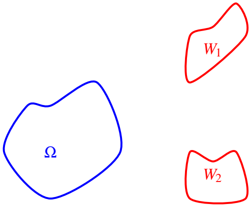
- Alessandrini's identity:

$$((\Lambda_{q_1} - \Lambda_{q_2}) f_1, f_2)_{W_2} = ((q_1 - q_2) u_1, u_2)_\Omega.$$

- Runge approximation ("All functions are locally s-harmonic up to a small error") [Dipiero-Savin-Valdinoci, Ghosh-Salo-Uhlmann, R.-Salo]; [Covi-R. '20].
- Robust; many extensions. [Ghosh-Lin-Xiao, Covi, Lai-Lin, Lai-Lin-R., Bhattacharya-Ghosh-Uhlmann, Cekić-Lin-R., Li, Covi-Mönkkönen-Railo-Uhlmann].



Heuristics



$$(-\Delta)^s u + qu = 0 \text{ in } \Omega,$$
$$u = f \text{ on } \Omega_e.$$

$$\Lambda_q : H^s(\Omega_e) \to H^{-s}(\Omega_e),$$
$$f \mapsto (-\Delta)^s u.$$

- Given info:

$$\Lambda_q f(x) = \int_{\Omega_e} k_q(x,y) f(y) dy$$

for all $x \in \Omega_e,\ f \in H^s(\Omega_e)$.

$\rightsquigarrow$ $2n$ degrees of freedom determined.

- Sought for info: $q : \Omega \to \mathbb{R}$.
  $\rightsquigarrow$ n unknown degrees of freedom.

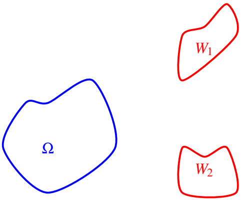
Problem always overdetermined in case of infinitely many measurements! In single measurement case still formally determined!

Single Measurement Uniqueness and Recovery

Theorem (Ghosh-R.-Salo-Uhlmann, '18)

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded open set. Let $s \in (0,1)$ and let $q \in C^0(\Omega)$. Let $W_1, W_2 \subset \Omega_e$ be open subsets of Ω_e. Given any $f \in \widetilde{H}^s(W_1) \setminus \{0\}$, the potential q is uniquely determined and can be reconstructed from the knowledge of $\Lambda_q f|_{W_2}$.

- **Single** measurement result (problem formally overdetermined for any dimension)!
- Extension to rougher potentials possible, $q \in L^\infty(\Omega)$ if $s \geq \frac{1}{4}$.
- Infinitely many measurement reconstruction [Harrach-Lin '17, '19].



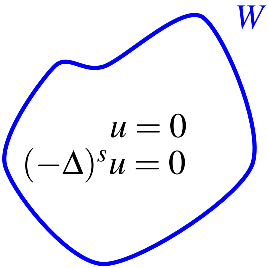
Unique Continuation for the Fractional Laplacian

Proposition

Let $u \in H^r(\mathbb{R}^n)$ and assume that on some open set W it holds

$$u = 0,\ (-\Delta)^s u = 0 \text{ on } W,$$

then $u \equiv 0$.



Ideas:

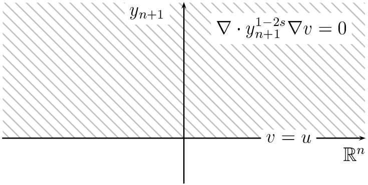
- [Fall-Felli '14]: Localization + frequency function approach.
- [R. '15, Ghosh-Salo-R.-Uhlmann '18]: Localization + Carleman.
- [García-Ferrero-R. '19]: Optimal dependences on metric.

Much stronger uniqueness than for Δ!
Holds for **quite general** fractional Schrödinger operators.

Unique Continuation: Localization

$$(-\Delta)^s u = \lim_{y_{n+1}\to 0} y_{n+1}^{1-2s}\partial_{n+1} v,$$

$$\nabla \cdot y_{n+1}^{1-2s} \nabla v = 0 \text{ in } \mathbb{R}^{n+1}_+,$$
$$v = u \text{ on } \mathbb{R}^n \times \{0\}.$$



Literature & Applications:

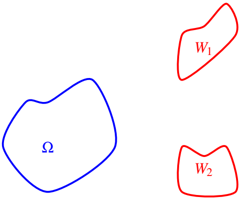
- [Caffarelli & Silvestre]: An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32 (2007).
- Applications: Quasi-geostrophic equation, free-boundary value problems, American options, relativistic Schrödinger equations.
- Interpretation of fractional Calderón problem as degenerate inverse Robin type problem [Covi-R. '20]; construction of CGOs for auxiliary problem.

Single Measurement Uniqueness and Recovery

Theorem (Ghosh-R.-Salo-Uhlmann, '18)

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded open set. Let $s \in (0,1)$ and let $q \in C^0(\Omega)$. Let $W_1, W_2 \subset \Omega_e$ be open subsets of Ω_e. Given any $f \in \widetilde{H}^s(W_1) \setminus \{0\}$, the potential q is uniquely determined and can be reconstructed from the knowledge of $\Lambda_q f|_{W_2}$.

- **Unique continuation:** Recovery of u from $u|_W = f, (-\Delta)^s u|_W = \Lambda_q(f)$.
- **Recovery algorithm**: $\Lambda_q(f), f \rightsquigarrow u$ and $q(x) := -\frac{(-\Delta)^s u(x)}{u(x)}$.
- **Unique continuation**: [R. '15], [Ghosh-R.-Salo-Uhlmann '18].



Single Measurement Uniqueness and Recovery

Theorem (Ghosh-R.-Salo-Uhlmann, '18)

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded open set. Let $s \in (0,1)$ and let $q \in C^0(\Omega)$. Let $W_1, W_2 \subset \Omega_e$ be open subsets of Ω_e. Given any $f \in \widetilde{H}^s(W_1) \setminus \{0\}$, the potential q is uniquely determined and can be reconstructed from the knowledge of $\Lambda_q f|_{W_2}$.

- **Recovery** of $v \in \widetilde{H}^s(\Omega)$ as $v = \lim\limits_{\alpha \to 0} v_\alpha$, where v_α is a solution to

$$v_\alpha = \operatorname{argmin}_{w \in \widetilde{H}^s(\Omega)} J_\alpha(w),$$
$$J_\alpha(w) := \|(-\Delta)^s w - (-\Delta)^s v\|^2_{H^{-s}(W)} + \|w - v\|^2_{H^s(W)} + \alpha \|w\|^2_{H^s(\mathbb{R}^n)}.$$

- **Recovery algorithm**: $\Lambda_q(f), f \rightsquigarrow u$ and $q(x) := -\frac{(-\Delta)^s u(x)}{u(x)}$.
- **Unique continuation**: [R. '15], [Ghosh-R.-Salo-Uhlmann '18].

Single Measurement Stability

Theorem (R., '20)

Let $\Omega, W \subset \mathbb{R}^n$, $n \geq 1$, *be bounded, open, non-empty Lipschitz sets with* $\overline{\Omega} \cap \overline{W} = \emptyset$. *Let* $f \in \tilde{H}^{s+\epsilon}(W) \setminus \{0\}$ *with* $\epsilon > 0$ *and* q_1, q_2 *with* $\text{supp}(q_j) \subset \Omega' \Subset \Omega$ *satisfy*

$$\|q_j\|_{C^{0,s}(\Omega)} \leq E < \infty, \ \frac{\|f\|_{H^s(W)}}{\|f\|_{L^2(W)}} \leq F < \infty.$$

Then, there exists a modulus of continuity $\omega : \mathbb{R}_+ \to \mathbb{R}$ *such that*

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leq \omega(\|\Lambda_{q_1}(f) - \Lambda_{q_2}(f)\|_{H^s(W)})$$

and $\omega(t) \leq C|\log(Ct)|^{-\sigma}$ *for* $t \in (0,1)$.

- **Quantify** steps from uniqueness argument!

Stability Ideas I: Quantitative Unique Continuation

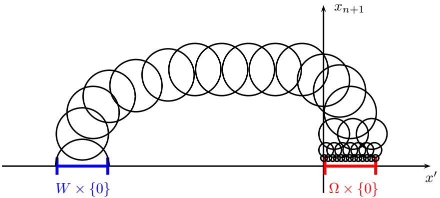
Aim:

- Quantitative recovery of $u_1|_\Omega, u_2|_\Omega$.
- Quantitative version of: $\Lambda_q(f), f \rightsquigarrow u$.

$$\|u_1 - u_2\|_{H^s(\Omega)} \leq C\|\Lambda_{q_1}(f) - \Lambda_{q_2}(f)\|_{H^{-s}(W)},$$
$$\|(-\Delta)^s u_1 - (-\Delta)^s u_2\|_{H^{-s}(\Omega)} \leq C\|\Lambda_{q_1}(f) - \Lambda_{q_2}(f)\|_{H^{-s}(W)}.$$

Here: $C = C(\|q_j\|_{L^\infty(\Omega)}, \|f\|_{H^{s+\epsilon}(W)})$; [R.-Salo '17, Ghosh-R.-Salo-Uhlmann '18].

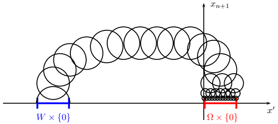
$$\nabla \cdot x_{n+1}^{1-2s} \nabla \tilde{u} = 0 \text{ in } \mathbb{R}^{n+1}_+,$$
$$\tilde{u} = u \text{ on } \mathbb{R}^n \times \{0\},$$
$$u = 0 \text{ on } W \times \{0\}.$$



Stability Ideas II: Boundary Doubling Estimates

Aim:

- Quantitative version of $q_j|_\Omega = \frac{(-\Delta)^s u_j}{u_j}$.
  Difficulty: control vanishing rate of u_j.



Proposition

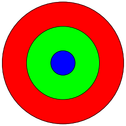
For $x_0 \in \Omega'$ and $r \in (0, r_0)$ with $r_0 \geq \text{dist}(\partial\Omega, \Omega')/10$ there exist $\beta > 0, C > 0$

$$\|u\|_{L^2(B_r'(x_0))} \geq C r^{\beta}.$$

Here: C in particular depends on F. Based on **bulk doubling** uses "compactness" of problem + oscillation control on f. [Sincich '10, Alessandrini-Sincich-Vessella '13].

$\exists C > 0$:

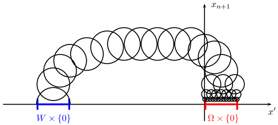
$$\|x_{n+1}^{\frac{1-2s}{2}} u\|_{L^2(B_{2r}^+(x_0))} \leq C \|x_{n+1}^{\frac{1-2s}{2}} u\|_{L^2(B_r^+(x_0))}.$$



Stability Ideas II: Boundary Doubling Estimates

Aim:

- Quantitative version of $q_j|_\Omega = \frac{(-\Delta)^s u_j}{u_j}$.
  Difficulty: control vanishing rate of u_j.



Proposition

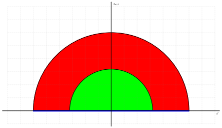
For $x_0 \in \Omega'$ and $r \in (0, r_0)$ with $r_0 \geq \text{dist}(\partial\Omega, \Omega')/10$ there exist $\beta > 0, C > 0$

$$\|u\|_{L^2(B_r'(x_0))} \geq C r^{\beta}.$$

Here: C in particular depends on F. Based on **bulk doubling** uses "compactness" of problem + oscillation control on f. [Sincich '10, Alessandrini-Sincich-Vessella '13].

$\exists C > 0$:

$$\|u\|_{L^2(B_{2r}'(x_0))} \leq C \|u\|_{L^2(B_r'(x_0))}.$$



Stability Estimates for General Nonlocal Operators

Consider

$$Tf(x) = \int_{\mathbb{R}^n} k(x,y) f(y) dy.$$

Aim: Use genuinely nonlocal arguments to derive stability estimates of the type

$$\|f\|_{L^2(I)} \leq C \frac{\|f\|_{H^1(I)}}{\left|\log\left(\frac{\|f\|_{L^2(I)}}{\|Tf\|_{L^2(J)}}\right)\right|^{\nu}}.$$

Example: The truncated Hilbert transform.

$$Tf(x) := H_{I,J} f(x) = \chi_J H(\chi_I f)(x),$$
$$\mathcal{F}(Hf)(\xi) = i\,\text{sgn}(\xi)\mathcal{F}f(\xi).$$

[Alaifari-Pierce-Steinerberger '14, R. '17, García-Ferrero-R. '20]

The Hilbert Transform I: The Moment Problem

$$C_c^\infty((0,1)) \ni f \mapsto (f_j)_{j\in\mathbb{N}},$$

$$f_j := \int_0^1 x^j f(x)dx.$$

- inverse problem ill-posed [Talenti].
- $\|f\|_{L^2(I)}^2 \leq e^{7N_f} \|\sum_{j=0}^{N_f} f_j x^j\|_{L^2(I)}$,

  $N_f := \frac{\|f\|_{H^1(I)}}{\|f\|_{L^2(I)}}$, $I \Subset (0,1)$.

For the Hilberttrafo [García-Ferrero-R. '20]:

$$H(f)(x) = \frac{1}{\pi x}\int_I \frac{f(y)}{1-\frac{y}{x}}dy = \frac{1}{\pi}\sum_{j=1}^\infty f_j x^{-j-1} = \frac{1}{\pi}\sum_{j=1}^\infty f_j z^j, \ x \notin \overline{I}.$$

$\rightsquigarrow \ \|f\|_{L^2(I)}^2 \ \leq \ e^{7N_f}\|\sum_{j=0}^{N_f} f_j z^j\|_{L^2(I)}^2 \ = \ e^{7N_f}\|\sum_{j=0}^{N_f} f_j x^{-j-1}\|_{L^2(I^{-1})}^2 \ = \ e^{7N_f}\|H(f)\|_{L^2(I^{-1})}^2$. $\rightsquigarrow$ quantitative analytic continuation [Vessella '99].

The Hilbert Transform I: The Moment Problem

$$C_c^{\infty}((0,1)) \ni f \mapsto (f_j)_{j\in\mathbb{N}},$$

$$f_j := \int_0^1 x^j f(x)dx.$$

- inverse problem ill-posed [Talenti].
- $\|f\|_{L^2(I)}^2 \leq e^{7N_f} \| \sum_{j=0}^{N_f} f_j x^j \|_{L^2(I)},$

$$N_f := \frac{\|f\|_{H^1(I)}}{\|f\|_{L^2(I)}},\ I \Subset (0,1).$$

For the Hilberttrafo [García-Ferrero-R. '20]:

$$H(f)(x) = \frac{1}{\pi x}\int_I \frac{f(y)}{1-\frac{y}{x}}dy = \frac{1}{\pi}\sum_{j=1}^{\infty} f_j x^{-j-1} = \frac{1}{\pi}\sum_{j=1}^{\infty} f_j z^j,\ x \notin \overline{I}.$$

$$\rightsquigarrow \|f\|_{L^2(I)}^2 \leq e^{7\frac{\|f\|_{H^1(I)}}{\|f\|_{L^2(I)}}} \|H(f)\|_{L^2(J)}^2.$$

The Hilbert Transform II: Branch-Cut Argument

For the Hilberttrafo [García-Ferrero-R. '20] (related qualitative ideas in [Liess, Isakov, R.-Salo]):

$$h_1(x) := Hf(x) + if(x) = 2i \int_0^\infty e^{ix\xi} \mathcal{F}f(\xi) d\xi,$$

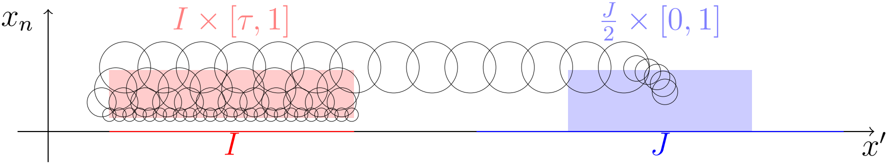
$$h_2(x) := Hf(x) - if(x) = -2i \int_{-\infty}^0 e^{ix\xi} \mathcal{F}f(\xi) d\xi, \ f \in C_c^\infty(I).$$

Analytic extensions into upper/lower complex half-planes (as functions of x) $\rightsquigarrow$ quantitative analytic continuation [Vessella '99].

$$\|h_j\|_{H^{-\frac{1}{2}}(I)} \leq C \frac{\|f\|_{H^1(I)}}{\left|\log\left(\frac{\|h_j\|_{H^{-1/2}(J)}}{\|f\|_{L^2(I)}}\right)\right|^\nu}.$$

$2if(x) = h_1(x) - h_2(x) \rightsquigarrow$ estimate for f in terms of data.

Remarks and Generalizations



Generalizations [García-Ferrero-R. '20]:

- higher dimensional analoga such as truncated Fourier and Laplace trafos,
- combinations local, nonlocal, pseudodiff operators

  $T(D) = |D_{x_n}|^s + L(D) + m(D')$, $L(D)$ local op, $m(D')$ pseudo in D'.
- Application: Quantitative Runge approximation for $L(D) := \sum_{j=1}^{n} (-\partial_{x_j}^2)^s + q$ for $q \in L^\infty$, $s \in (0,1)$.

Summary and Outlook

Results:

- Strong properties of non-local equations in inverse problems.
- Quantitative unique continuation important in stability analysis.
- Combines duality with ideas from control theory.

Questions:

- Implications for $s = 1$?
- More general operators?
- Anisotropic problem?