Coherent Acousto-Optic Imaging

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Joint work with Francis Chung (Kentucky) and
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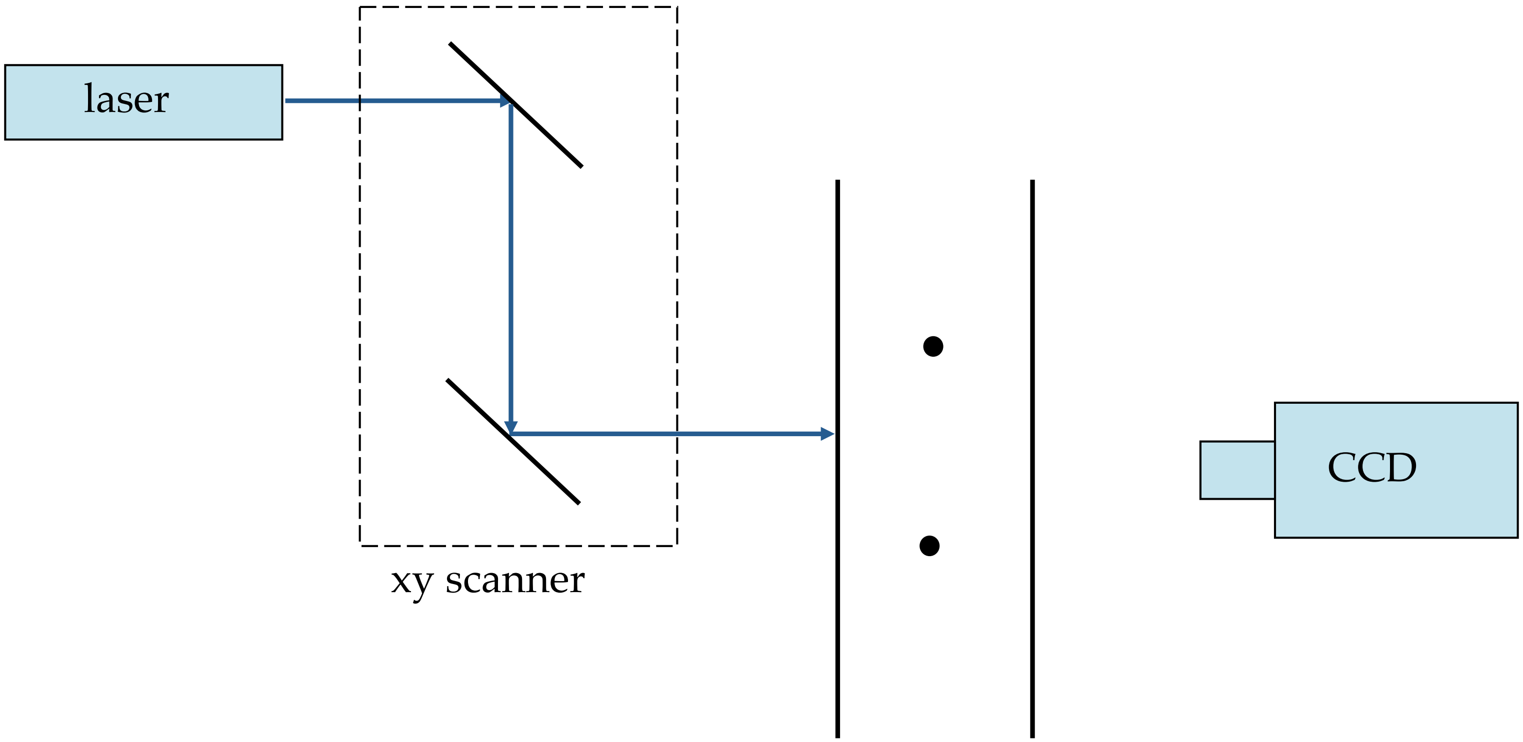
Optical tomography



$10^8 - 10^{10}$ source-detector pairs

Inverse problem

The energy density u of multiply-scattered light in an absorbing medium obeys the diffusion equation

$$\begin{aligned} -\Delta u + \sigma(\mathbf{x})u &= 0 \quad \text{in} \quad \Omega \ , \\ u &= g \quad \text{on} \quad \partial\Omega \ . \end{aligned}$$

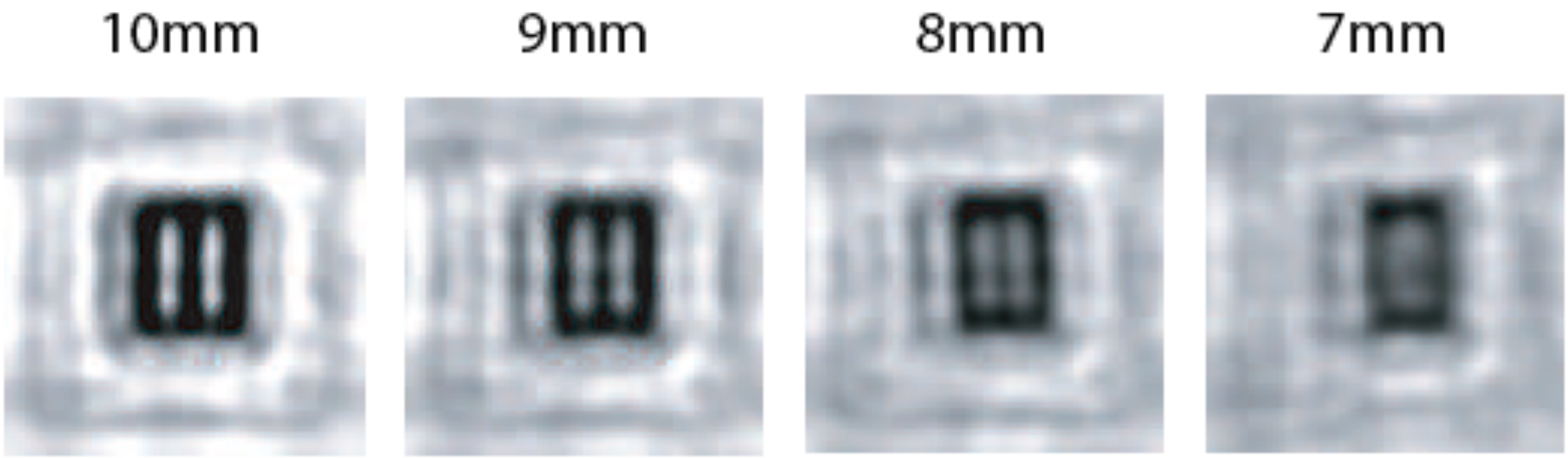
The **inverse problem** is to recover the absorption σ from the Dirichlet to Neumann map

$$\Lambda_\sigma : g \mapsto \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} \ .$$

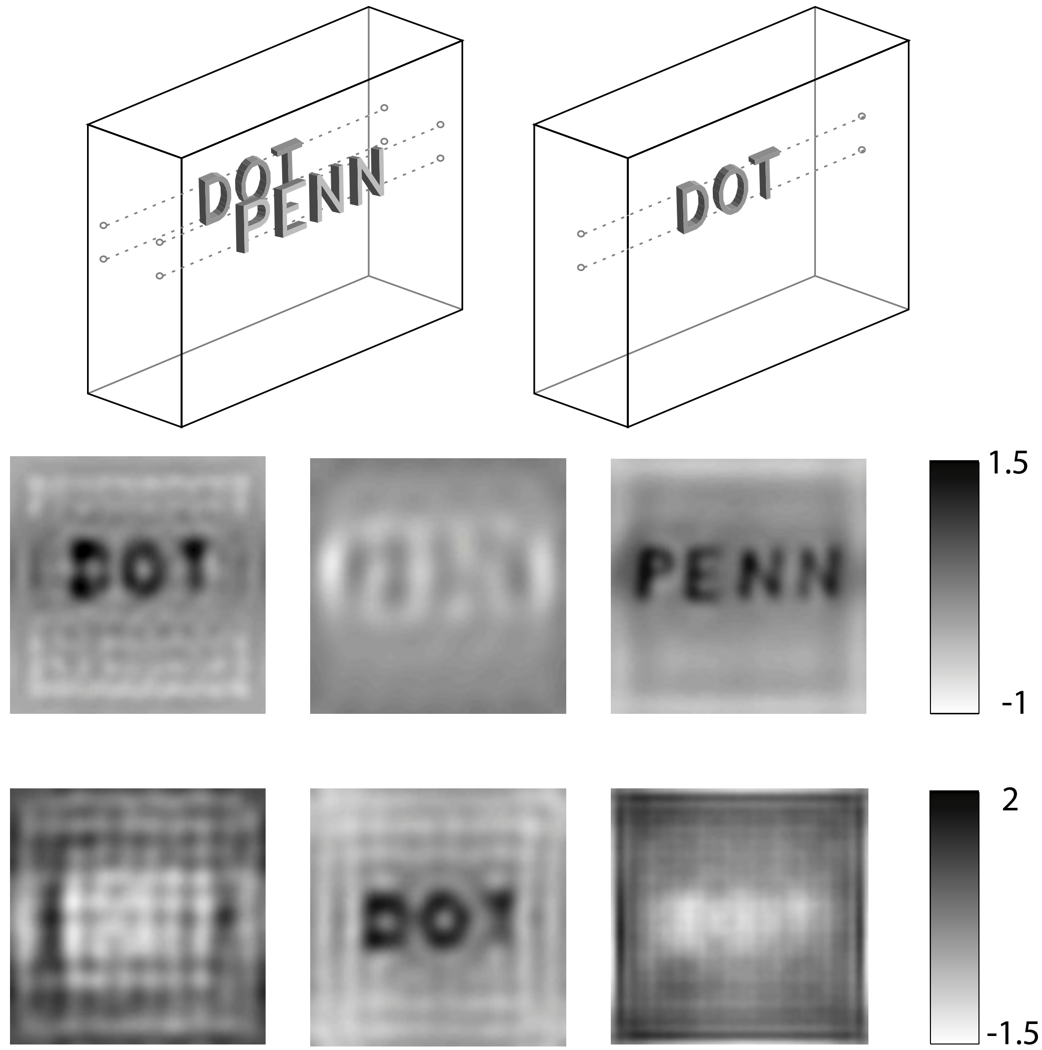
This problem is severely ill-posed with logarithmic stability, due to exponential decay of the energy density.

Even with large data sets and fast algorithms, there is a fundamental limit to what can be done.

Resolution and bar targets



Proxy for anatomy



Acousto-optic imaging

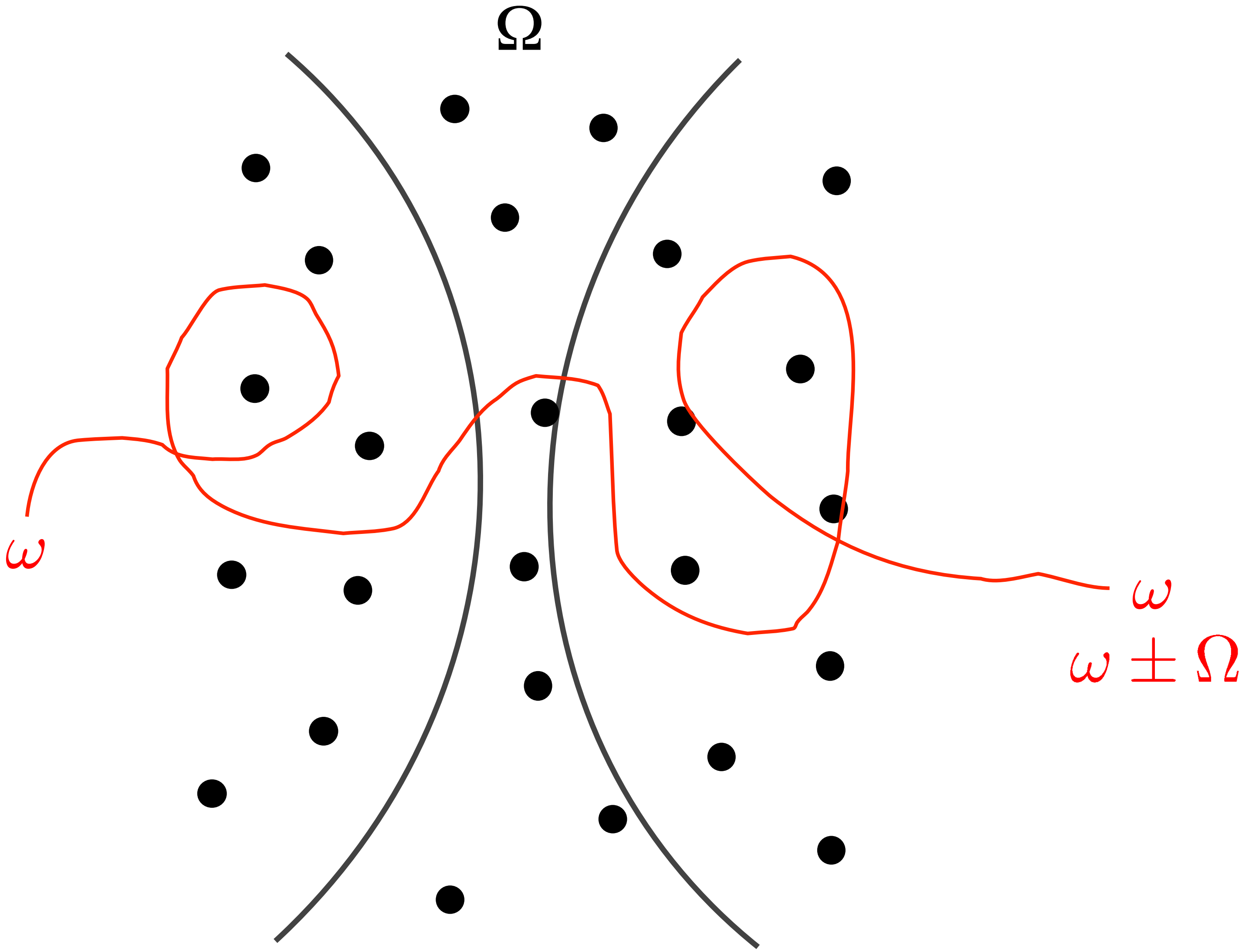
- To overcome the problem of low image resolution requires a new physical idea.

- Internal measurements of the optical field directly determine the coefficient σ.

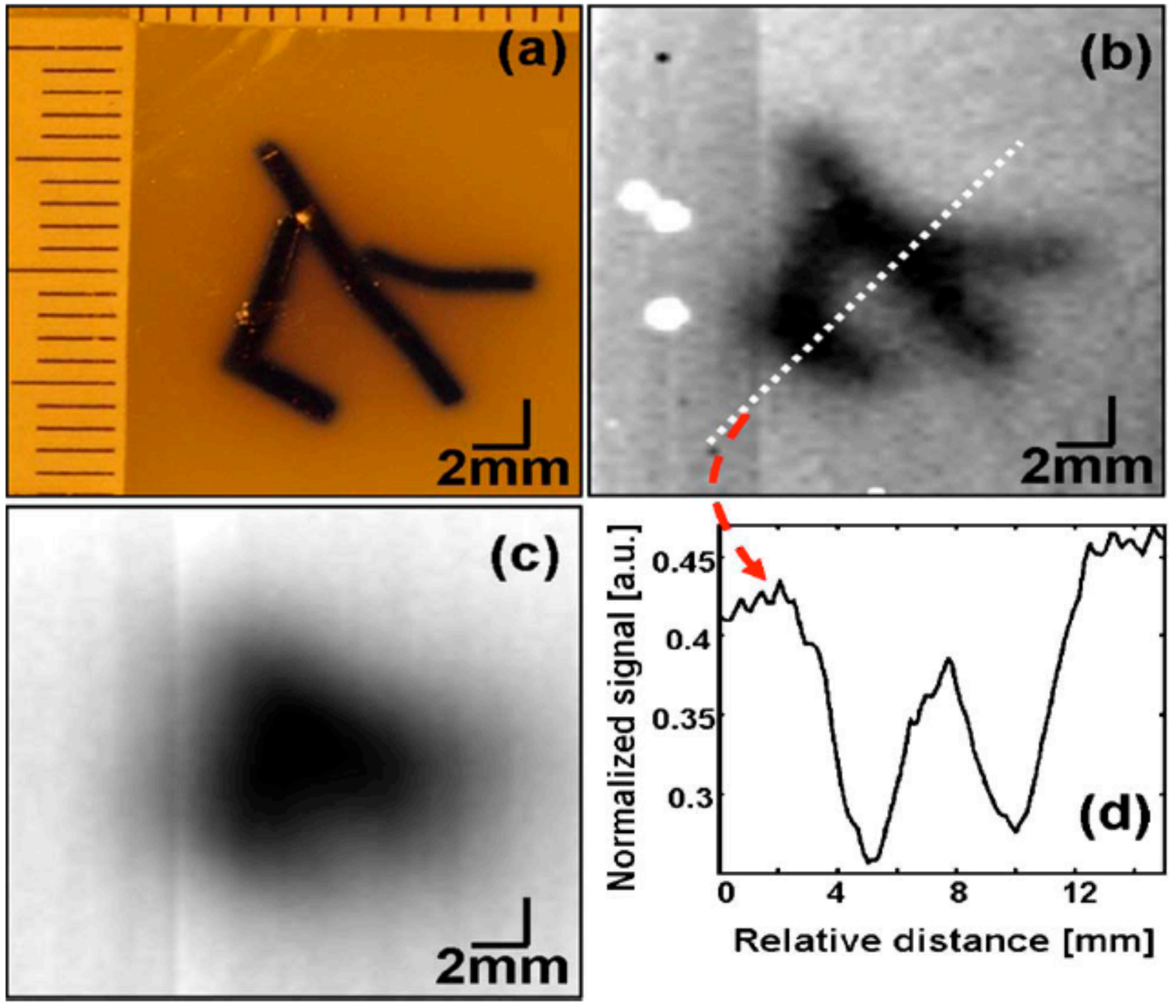
- Replace inverse boundary value problems by inverse problems with internal data.

- Internal data obtained from control of boundary measurements by an acoustic wave.

Acousto-optic effect



Acousto-optic imaging



Data of Lihong Wang (Cal Tech)

Inverse problem

Acousto-optic imaging utilizes two interacting wave fields. In some sense, it uses the best of both waves. The acoustic wave provides high spatial resolution while the optical field is sensitive to the contrast of interest.

The inverse problem consists of two steps.

1. Recover an internal functional of the coefficients from acousto-optic measurements. The internal functional is defined at every point of the medium and serves as a proxy for measurements of the optical field within the medium.

2. Reconstruct the coefficients from the internal functional.

Coherent vs incoherent acousto-optics

In **incoherent** acousto-optics an intensity measurement of the scattered light at the fundamental frequency is performed. This takes advantage of the separation of scales $\Omega \ll \omega$ so that the medium changes slowly (the measurement is effectively instantaneous). A direct image of the medium is not formed. However, an inverse problem can be formulated in this setting.

In **coherent** acousto-optics an interferometric measurement of the scattered light at the shifted frequencies $\omega \pm \Omega$ is performed. A direct image of the medium is formed.

Incoherent inverse problem

The energy density u obeys the diffusion equation

$$\begin{aligned} -\nabla \cdot D(\mathbf{x})\nabla u + \sigma(\mathbf{x})u &= 0 \quad \text{in} \quad \Omega \ , \\ u &= g \quad \text{on} \quad \partial\Omega \ . \end{aligned}$$

Here D is the diffusion coefficient and σ is the attenuation coefficient.

It can be shown [Bal-S] that the internal functional is of the form

$$H(\mathbf{x}) = D(\mathbf{x})|\nabla u|^2 - \sigma(\mathbf{x})u^2 \ , \quad \mathbf{x} \in \Omega \ .$$

Making use of sufficiently many boundary sources, leads to a nonlinear elliptic system.

Related work

- Bal-Chung-S

  — inverse diffusion

  — inverse transport

  — inverse source problem (diffusion and transport)

- Ammari-Garnier-Seppecher

  — inverse diffusion

  — iterative reconstruction method

- Arridge-Powell

  — optimization method

Outline

- First-principles theory of coherent acousto-optic effect

  — mechanics

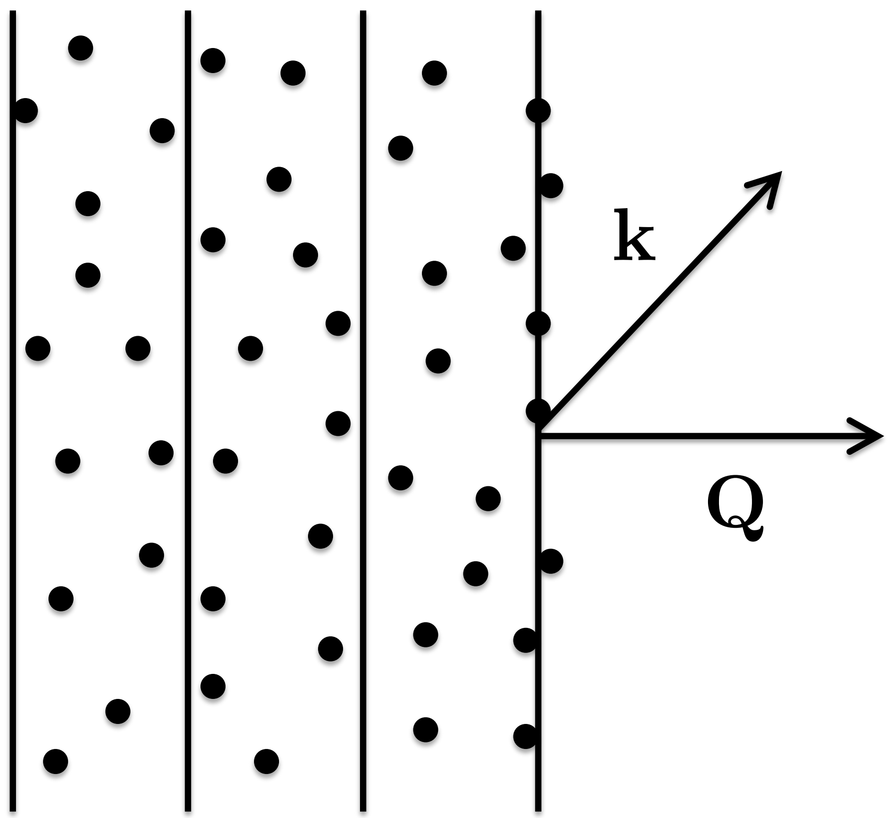
  — light propagation

- Acousto-optics in random media

  — radiative transport and diffusion

- Inverse problem

Acousto-optic effect



Mechanics

We consider an acoustic wave propagating in a medium consisting of identical particles suspended in a fluid. If the amplitude of the acoustic wave is sufficiently small, the particles will oscillate about their equilibrium positions. It is then possible to treat the motion of each particle as independent, neglecting hydrodynamic interactions.

The equation of motion of a single particle is of the form

$$\varrho \frac{d\mathbf{u}}{dt} = -\nabla p + \frac{4\pi a \eta}{V}(\mathbf{v} - \mathbf{u}) \ .$$

Here $\mathbf{u}$ denotes the velocity of the particle, p is the pressure, $\mathbf{v}$ is the velocity field in the fluid, η is the viscosity, a is the radius of the particle, ϱ is its mass density and V is its volume. The second term is due to the relative motion of the particles (Stokes-Einstein).

Consider a standing time-harmonic acoustic wave with pressure

$$p(\mathbf{x},t) = p_0 \cos(\Omega t) \cos(\mathbf{Q} \cdot \mathbf{x}) \ ,$$

where p_0 is the amplitude of the wave, Ω is its frequency and $\mathbf{Q}$ is the wave vector. Here we have assumed that the speed of sound c_s is constant with $Q = \Omega / c_s$.

The number density of particles is modulated by the acoustic wave. Let $\mathbf{x}_1, \ldots, \mathbf{x}_N$ denote the positions of the particles and

$$\rho(\mathbf{x},t) = \sum_{j=1}^{N} \delta(\mathbf{x} - \mathbf{x}_j(t))$$

be their density. Since each particle is independent, it follows from integration of the equations of motion that ρ is modulated according to

$$\rho(\mathbf{x},t) = \rho_0(\mathbf{x}) \left[1 + \delta \cos(\Omega t) \cos(\mathbf{Q} \cdot \mathbf{x})\right] \ ,$$

where ρ_0 is the number density of the particles in the absence of the acoustic wave and $\delta = p_0/(\rho c_s^2) \ll 1$.

Propagation of light

The optical field $u(\mathbf{x}, t)$ is taken to obey the wave equation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\varepsilon(\mathbf{x}, t)u) = \Delta u \ .$$

The dielectric permittivity ε has contributions from the fluid and the particles:

$$\varepsilon(\mathbf{x}, t) = \varepsilon_0(\mathbf{x}, t) + 4\pi\eta(\mathbf{x}, t) \ ,$$

where ε_0 is the permittivity of the fluid and η is the dielectric susceptibility of the particles. The permittivity of the fluid is acoustically modulated due to Brillouin scattering and is given by

$$\varepsilon_0(\mathbf{x}, t) = \varepsilon_0 \left[1 + \delta\gamma \cos(\Omega t) \cos(\mathbf{Q} \cdot \mathbf{x})\right] \ ,$$

where ε_0 is constant and γ is the elasto-optical constant. The susceptibility is proportional to ρ and is given by

$$\eta(\mathbf{x}, t) = \alpha\rho_0(\mathbf{x}) \left[1 + \delta \cos(\Omega t) \cos(\mathbf{Q} \cdot \mathbf{x})\right] \ ,$$

where α is the polarizability of a single particle.

Mode decomposition

We suppose that the incident optical field is monochromatic with frequency ω and decompose u in harmonics of the acoustic frequency:

$$u(\mathbf{x},t) = \sum_{n=-\infty}^{\infty} e^{-i(\omega+n\Omega)t} u_n(\mathbf{x}) \ .$$

The frequency components u_n obey the system of coupled Helmholtz equations

$$\Delta u_n + k_n^2 \left(\varepsilon_0 + 4\pi\eta(\mathbf{x})\right) u_n = -\frac{\delta k_n^2}{2} \left(\gamma\varepsilon_0 + 4\pi\eta(\mathbf{x})\right) \cos(\mathbf{Q}\cdot\mathbf{x}) \left(u_{n-1} + u_{n+1}\right)$$

where $k_n = (\omega + n\Omega)/c$ and $\eta(\mathbf{x}) = \alpha\rho_0(\mathbf{x})$. Note that if $u_0 = O(1)$ then $u_n = O(\delta^n)$. We close the equations for the lowest order modes according to

$$\begin{aligned}
\Delta u_0 + k_0^2 \left(\varepsilon_0 + 4\pi\eta(\mathbf{x})\right) u_0 &= 0 \ , \\
\Delta u_1 + k_1^2 \left(\varepsilon_0 + 4\pi\eta(\mathbf{x})\right) u_1 &= -\frac{\delta k_1^2}{2} \left(\gamma\varepsilon_0 + 4\pi\eta(\mathbf{x})\right) \cos(\mathbf{Q}\cdot\mathbf{x}) u_0 \ , \\
\Delta u_{-1} + k_{-1}^2 \left(\varepsilon_0 + 4\pi\eta(\mathbf{x})\right) u_{-1} &= -\frac{\delta k_{-1}^2}{2} \left(\gamma\varepsilon_0 + 4\pi\eta(\mathbf{x})\right) \cos(\mathbf{Q}\cdot\mathbf{x}) u_0 \ .
\end{aligned}$$

Homogeneous medium

Consider the case of a homogeneous medium without scatterers. The fundamental mode u_0 acts as a source of the first harmonics $u_{\pm 1}$. If the field u_0 is a unit-amplitude plane wave:

$$u_0 = e^{i\mathbf{k}\cdot\mathbf{x}} \ , \quad k = \sqrt{\varepsilon_0} k_0 \ ,$$

we find that u_1 is given by

$$u_1 = \frac{1}{4}\delta k'^2 \left[\frac{1}{(\mathbf{Q}+\mathbf{k})^2 - k'^2} e^{i(\mathbf{k}+\mathbf{Q})\cdot\mathbf{x}} + \frac{1}{(\mathbf{Q}-\mathbf{k})^2 - k'^2} e^{i(\mathbf{k}-\mathbf{Q})\cdot\mathbf{x}} \right] ,$$

where $k' = \sqrt{\varepsilon_0} k_1$. For fixed $\mathbf{Q}$, there is a resonance if the incident wave vector $\mathbf{k}$ obeys the condition

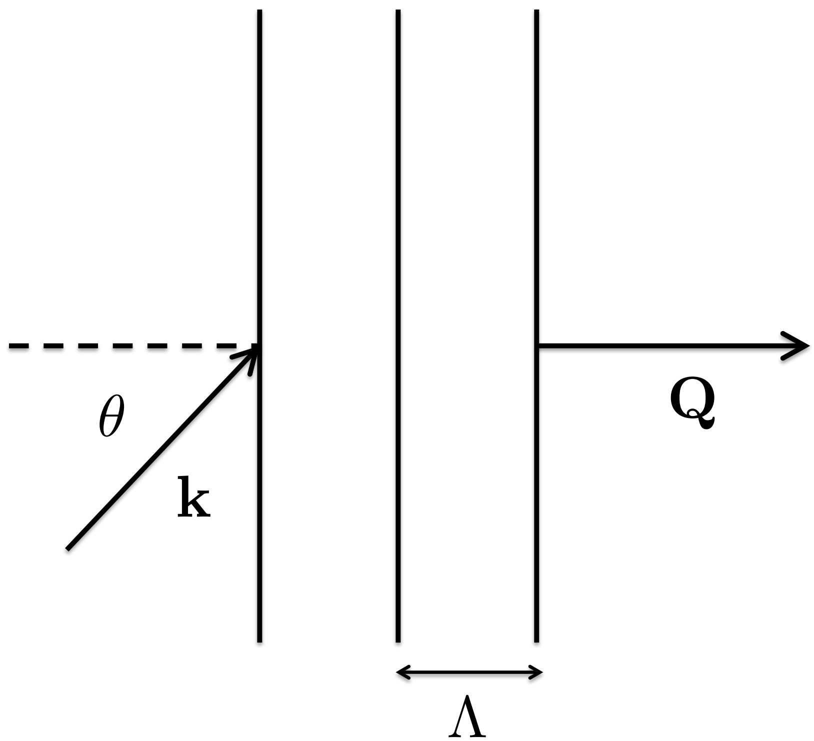
$$|\mathbf{Q} \pm \mathbf{k}| = k' \ .$$

We note that the presence of absorption in the fluid prohibits the formation of a resonance.

Bragg condition

The direction of $\mathbf{k}$ is set by

$$\sin\theta = \pm\frac{\lambda}{2\Lambda} \ .$$



Acousto-optics in random media

We now discuss radiative transport theory for the acousto-optic effect.

We consider the propagation of light in random media.

$$\begin{aligned} \Delta u_0 + k_0^2\,(\varepsilon_0 + 4\pi\eta(\mathbf{x}))\,u_0 &= 0\ , \\ \Delta u_1 + k_1^2\,(\varepsilon_0 + 4\pi\eta(\mathbf{x}))\,u_1 &= -\frac{\delta k_1^2}{2}\,(\gamma\varepsilon_0 + 4\pi\eta(\mathbf{x}))\cos(\mathbf{Q}\cdot\mathbf{x})u_0\ . \end{aligned}$$

The random medium is statistically homogeneous and isotropic with correlations

$$\langle\eta\rangle = 0\ , \qquad \langle\eta(\mathbf{x})\eta(\mathbf{x}')\rangle = C(\mathbf{x}-\mathbf{x}')\ ,$$

where $\langle\cdots\rangle$ denotes statistical averaging.

The modes u_0 and u_1 oscillate on the scale of the wavelength. We are interested in high-frequency asymptotics. For such solutions, the propagation distance is long compared to the wavelength and η is slowly varying.

We introduce a slow space coordinate $\mathbf{x} \to \mathbf{x}/\epsilon$, where ϵ is small. The rescaled amplitudes $u_\epsilon(\mathbf{x}) = u_0(\mathbf{x}/\epsilon)$ and $v_\epsilon(\mathbf{x}) = u_1(\mathbf{x}/\epsilon)$ satisfy

$$\epsilon^2 \Delta u_\epsilon + k_0^2 \left(\varepsilon_0 + 4\pi\sqrt{\epsilon}\eta(\mathbf{x}/\epsilon)\right) u_\epsilon = 0 \ ,$$

$$\epsilon^2 \Delta v_\epsilon + k_1^2 \left(\varepsilon_0 + 4\pi\sqrt{\epsilon}\eta(\mathbf{x}/\epsilon)\right) v_\epsilon = -\frac{\delta k_1^2}{2} \left(\gamma\varepsilon_0 + 4\pi\sqrt{\epsilon}\eta(\mathbf{x}/\epsilon)\right) \cos(\mathbf{Q}\cdot\mathbf{x}) u_\epsilon \ .$$

We consider the high-frequency limit $\epsilon \to 0$ and rescale η so that the randomness is sufficiently weak and take C to be $O(\epsilon)$.

Wigner transform

The approach to radiative transport is through the Wigner transform

$$W_\epsilon(\mathbf{x},\mathbf{k}) = \int d^3x' e^{i\mathbf{k}\cdot\mathbf{x}'}\phi_\epsilon(\mathbf{x}-\epsilon\mathbf{x}'/2)\phi_\epsilon^*(x+\epsilon\mathbf{x}'/2)\ ,$$

where $\phi_\epsilon = (u_\epsilon, v_\epsilon)$. W_ϵ plays the role of an angularly resolved phase-space energy density. In the high-frequency limit, W_ϵ can be interpreted as the specific intensity in radiative transport theory.

The diagonal elements of W_ϵ are real-valued, but not generally non-negative. However, in the high-frequency limit $\epsilon \to 0$, W_ϵ becomes nonnegative.

The high-frequency, weak disorder regime is precisely the setting in which radiative transport theory holds.

We find that $I = \langle W \rangle$ obeys the system of radiative transport equations

$$\begin{aligned}
\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} I_{00} + \mu_s I_{00} - \mu_s L I_{00} &= 0 \ , \\
\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} I_{01} + \mu_s I_{01} - \mu_s L I_{01} &= \frac{1}{4} \gamma \varepsilon_0 k_0 \cos(\mathbf{Q} \cdot \mathbf{x}) I_{00} \ , \\
\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} I_{11} + \mu_s I_{11} - \mu_s L I_{11} &= \frac{1}{2} \gamma \varepsilon_0 k_0 \cos(\mathbf{Q} \cdot \mathbf{x}) I_{01} \ ,
\end{aligned}$$

where

$$L I(\mathbf{x}, \hat{\mathbf{k}}) = \int f(\hat{\mathbf{k}}, \hat{\mathbf{k}}') I(\mathbf{x}, \hat{\mathbf{k}}') d\hat{\mathbf{k}}' \ .$$

Here the scattering coefficient μ_s and the scattering kernel f are defined by

$$\mu_s = k_0^4 \int \tilde{C}(k_0(\hat{\mathbf{k}} - \hat{\mathbf{k}}')) d\hat{\mathbf{k}}', \quad f(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = \frac{\tilde{C}(k_0(\hat{\mathbf{k}} - \hat{\mathbf{k}}'))}{\int \tilde{C}(k_0(\hat{\mathbf{k}} - \hat{\mathbf{k}}')) d\hat{\mathbf{k}}'} \ .$$

This is a long story. The quantity I_{00} is the specific intensity of light at the fundamental frequency. Similarly, I_{11} is the specific intensity of the first harmonic. Note that I_{01} is related to correlations of the modes u_0 and u_1 and can be obtained in an interferometric experiment.

Key ingredients

- We consider W_ϵ in the high-frequency limit $\epsilon \to 0$ and introduce a multiscale expansion of the form

$$W_\epsilon(\mathbf{x},\mathbf{k}) = W_0(\mathbf{x},\mathbf{k}) + \sqrt{\epsilon} W_1(\mathbf{x},\mathbf{X},\mathbf{k}) + \epsilon \mathbf{W_2}(\mathbf{x},\mathbf{X},\mathbf{k}) + \cdots \ ,$$

where $\mathbf{X} = \mathbf{x}/\epsilon$ is a fast variable and W_0 is taken to be deterministic.

- By separating terms of order $O(1)$, $O(\sqrt{\epsilon})$ and $O(\epsilon)$, we obtain a hierarchy of kinetic equations. By averaging over η and introducing a suitable closure, we get the required radiative transport equations.

Diffusion limit

In the multiple-scattering regime with weak absorption ($\mu_a \ll \mu_s$), the specific intensity is slowly varying in $\widehat{\mathbf{k}}$:

$$I_{00}(\mathbf{x}, \widehat{\mathbf{k}}) = \frac{1}{4\pi}\left(U_{00} - \ell^* \widehat{\mathbf{k}} \cdot \nabla U_{00}\right) \ ,$$
$$I_{01}(\mathbf{x}, \widehat{\mathbf{k}}) = \frac{1}{4\pi}\left(U_{01} - \ell^* \widehat{\mathbf{k}} \cdot \nabla U_{01}\right) \ ,$$
$$I_{11}(\mathbf{x}, \widehat{\mathbf{k}}) = \frac{1}{4\pi}\left(U_{11} - \ell^* \widehat{\mathbf{k}} \cdot \nabla U_{11}\right) \ .$$

The energy densities obey the diffusion equations

$$-D\Delta U_{00} + \sigma U_{00} = 0 \ ,$$
$$-D\Delta U_{01} + \sigma U_{01} = \frac{1}{4}\gamma \varepsilon_0 \omega \cos(\mathbf{Q} \cdot \mathbf{x}) U_{00} \ ,$$
$$-D\Delta U_{11} + \sigma U_{11} = \frac{1}{2}\gamma \varepsilon_0 \omega \cos(\mathbf{Q} \cdot \mathbf{x}) U_{01} \ ,$$

The diffusion coefficient is defined by $D = 1/3c\ell^*$, where the transport mean free path ℓ^* is given by $\ell^* = 1/\left[(1-g)\mu_s + \mu_a\right]$ and $\sigma = c\mu_a$.

Inverse problem

Consider the forward problem

$$-\nabla \cdot D(\mathbf{x})\nabla U_{00} + \sigma(\mathbf{x})U_{00} = 0 \quad \text{in} \quad \Omega \ ,$$
$$-\nabla \cdot D(\mathbf{x})\nabla U_{01} + \sigma(\mathbf{x})U_{01} = A\cos(\mathbf{Q}\cdot\mathbf{x} + \varphi)U_{00} \quad \text{in} \quad \Omega \ ,$$

together with the boundary conditions

$$U_{00} + \ell\hat{\mathbf{n}}\cdot\nabla U_{00} = g \quad \text{on} \quad \partial\Omega \ ,$$
$$U_{01} + \ell\hat{\mathbf{n}}\cdot\nabla U_{01} = 0 \quad \text{on} \quad \partial\Omega \ .$$

The inverse problem is to recover σ and D from boundary measurements of U_{01}: $\Lambda_{(\sigma,D)} : (\mathbf{Q},\varphi) \mapsto \partial_n U_{01}|_{\partial\Omega}$, where the source g is fixed.

Internal functional

The identity

$$\frac{1}{Al}\int_{\partial\Omega} D(\mathbf{x})U_{01}(\mathbf{x})g(\mathbf{x})d\mathbf{x} = \int_{\Omega} U_{00}^2(\mathbf{x})\cos(\mathbf{Q}\cdot\mathbf{x}+\varphi)d\mathbf{x} \ ,$$

follows from an integration by parts and use of the boundary conditions, we obtain

Since g, D and σ are nonnegative, it follows that the internal functional

$$H(\mathbf{x}) = U_{00}(\mathbf{x}) \ , \quad x \in \Omega$$

can be determined from the data by Fourier inversion.

The above should be compared to the incoherent internal functional

$$H(\mathbf{x}) = D(\mathbf{x})|\nabla U_{00}|^2 - \sigma(\mathbf{x})U_{00}^2 \ , \quad x \in \Omega \ ,$$

which leads to a nonlinear elliptic system.

Inversion formulas

If the diffusion coefficient D is constant, it follows that

$$\sigma(\mathbf{x}) = D\frac{\Delta H(\mathbf{x})}{H(\mathbf{x})} \ .$$

Here we utilize a single boundary source. Note that H cannot vanish.

If D is not constant, we can recover (σ, D) from a pair of boundary sources. Suppose H_1 and H_2 are internal functionals corresponding to two well chosen boundary sources. Then D can be obtained by solving the transport equation

$$\begin{aligned} \mathbf{A} \cdot \nabla D + BD = 0 \quad \text{in} \quad \Omega \ , \\ D = D_0 \quad \text{on} \quad \partial\Omega \ , \end{aligned}$$

where

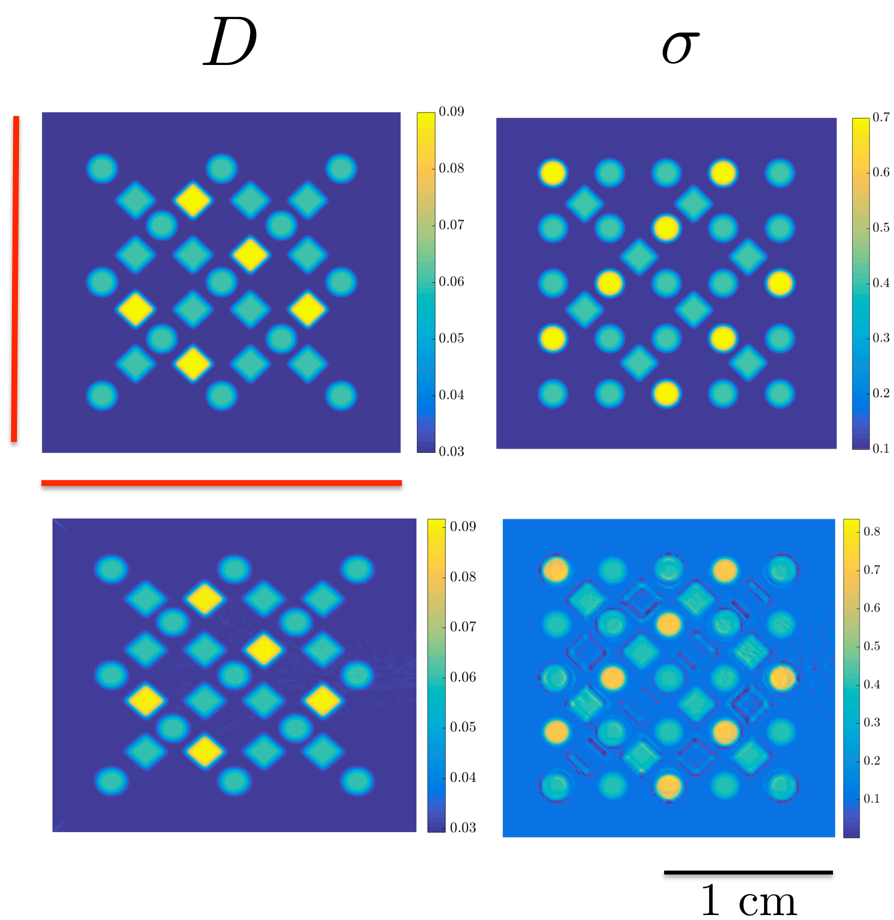
$$\mathbf{A} = \nabla H_1/H_2 \ , \quad B = H_1/H_2 \ .$$

Once D is known, we can find σ from the formula

$$\sigma(\mathbf{x}) = \frac{\nabla \cdot D(\mathbf{x}) \nabla H(\mathbf{x})}{H(\mathbf{x})} \ .$$

It can be shown that (σ, D) can be recovered with Holder stability. See Bal-Uhlmann (2013).

Reconstructions



Summary

- We have developed the theory of the acousto-optic effect in random media.

- We have studied the inverse problem of coherent acousto-optic imaging in the diffusion limit.

- The coherent inverse problem is considerably simpler than the corresponding incoherent problem. This comes at the expense of some increase in experimental complexity.

- We have also investigated the corresponding inverse transport problem.

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THANK YOU!