Online Pre-Conference
Inverse Problems and Nonlinearity

40 Years of Calderón's Problem

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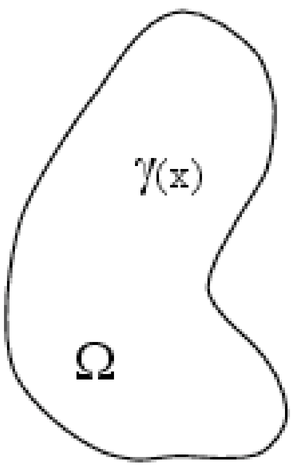
Alberto P. Calderón (1920-1997)

Highlights of Calderón's Career

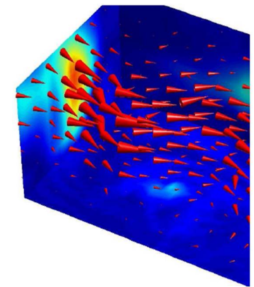
- Calderón Zygmund Singular Integral Theory
- Uniqueness for the Cauchy Problem
- The Complex Interpolation Method
- Calderón's Reproducing Formula
- The Calderón Projector
- Calderón-Vaillancourt Theorem
- L^2 boundedness of the Cauchy Integral on Lipschitz Curves (with small constant)
- Inverse Boundary Problem: On an inverse boundary value problem, in Seminar on Numerical Analysis and its Applications to Continuum Physics, Río de Janeiro, 1980.

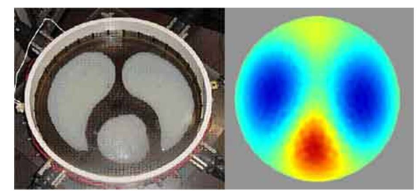
CALDERÓN'S PROBLEM and EIT



$\Omega \subset \mathbb{R}^n$

$(n \geq 2)$





Can one determine the electrical conductivity of $\Omega, \gamma(x)$, by making voltage and current measurements at the boundary?
(Calderón; Geophysical prospection)

Early breast cancer detection

Normal breast tissue	0.3 mho
Cancerous breast tumor	2.0 mho

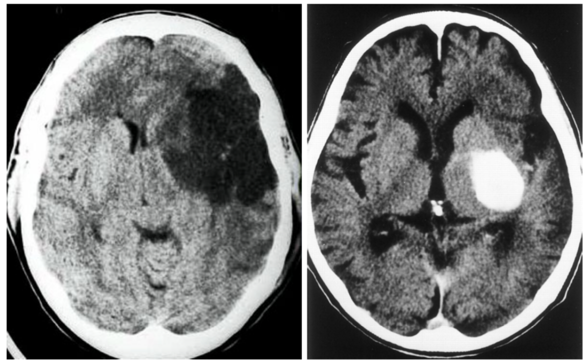
REMINISCENCIA DE MI VIDA MATEMÁTICA

Speech at Universidad Autónoma de Madrid accepting the 'Doctor Honoris Causa':

> *My work at "Yacimientos Petrolíferos Fiscales" (YPF) was very interesting, but I was not well treated, otherwise I would have stayed there.*

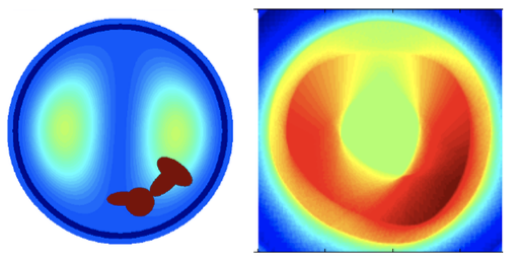
Imaging Stroke with EIT

Ischemic stroke:
low conductivity.
CT image from
Jansen 2008

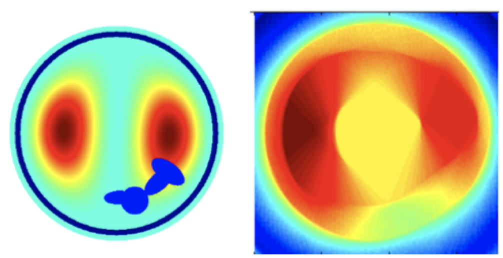


Hemorrhagic stroke:
high conductivity.
CT image from
Nakano *et al.* 2001

Same symptoms in both cases!



Simulated hemorrhage in the brain: higher conductivity because of excess blood.
Left: original, right: reconstruction



Simulated ischemic stroke: lower conductivity resulting from a clot blocking the flow of blood.
Left: original, right: reconstruction

Greenleaf, Lassas, Santacesaria, Siltanen–U, 2018

Stroke Imaging (Greenleaf–Lassas–Santacesaria–U, 2018)

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CALDERÓN'S PROBLEM (EIT)

Consider a body $\Omega \subset \mathbb{R}^n$. An electrical potential $u(x)$ causes the current

$$\boxed{I(x) = \gamma(x)\nabla u}$$

The conductivity $\gamma(x)$ can be isotropic, that is, scalar, or anisotropic, that is, a matrix valued function. If the current has no sources or sinks, we have

$$\boxed{\operatorname{div}(\gamma(x)\nabla u) = 0} \text{ in } \Omega$$

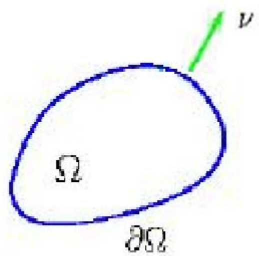
DN Map

$$\text{div}(\gamma(x)\nabla u(x)) = 0$$
$$u\big|_{\partial\Omega} = f$$

γ = conductivity, $\gamma \geq c > 0$

f = voltage potential at $\partial\Omega$

Current flux at $\partial\Omega$ = $(\nu \cdot \gamma\nabla u)\big|_{\partial\Omega}$ were ν is the unit outer normal.



Information is encoded in map

$$\Lambda_\gamma(f) = \nu \cdot \gamma\nabla u\big|_{\partial\Omega}$$

Calderón's inverse problem: Does Λ_γ determine γ?

Λ_γ = Dirichlet-to-Neumann map

Calderón's Paper

Linearized problem at $\gamma = 1$:

$$\boxed{\int_\Omega h \nabla u \cdot \nabla v dx} \qquad \text{data} \quad \forall\ \Delta u = \Delta v = 0.$$

Can we recover h?

$$\begin{array}{l} u = e^{x\cdot\rho} \\ v = e^{-x\cdot\overline{\rho}} \end{array}, \qquad \rho \in \mathbb{C}^n,\ \rho\cdot\rho = 0.$$

$$\rho = \frac{\eta - i\xi}{2}, \qquad \rho\cdot\rho = 0 \ \Leftrightarrow\ |\eta| = |\xi|, \eta\cdot\xi = 0.$$

$$\boxed{|\xi|^2 \int_\Omega h e^{-ix\cdot\xi} dx} \qquad \text{known}$$

we can recover $\boxed{\widehat{\chi_\Omega h}(\xi)}$, therefore h on Ω.

Boundary Determination

Theorem (Kohn-Vogelius, 1984)

Assume $\gamma \in C^\infty(\overline{\Omega})$. From $\boxed{\Lambda_\gamma}$ we can determine $\boxed{\partial^\alpha \gamma\big|_{\partial\Omega}}$, $\forall\, \alpha$.

Proof (Sylvester-U, 1988, Lee-U, 1989)

$\boxed{\Lambda_\gamma}$ is a pseudodifferential operator of order 1 (Calderón).

$$\boxed{\Lambda_\gamma f(x') = \int e^{ix'\cdot\xi'} \lambda_\gamma(x',\xi') \widehat{f}(\xi') d\xi'}$$

$$\boxed{\lambda_\gamma(x',\xi') = \gamma(0,x')|\xi'| + a_0(x',\xi') + \cdots + a_j(x',\xi') + \cdots}$$

with $a_j(x',\xi')$ pos. homogeneous of degree $-j$ in ξ':

$$a_j(x',\lambda\xi') = \lambda^{-j} a_j(x',\xi'), \quad \lambda > 0.$$

$$\gamma(x') = \lim_{|\xi'|\to\infty} \frac{1}{|\xi'|} e^{ix'\cdot\xi'} \Lambda_\gamma(e^{-ix'\cdot\xi'}).$$

$\boxed{\textbf{Result}}$ From a_j, we can determine $\frac{\partial^j \gamma}{\partial \nu^j}\Big|_{x^n=0}$.

Uniqueness

Theorem $n \geq 3$ (Sylvester-U, 1987)

$$\gamma \in C^2(\overline{\Omega}), \qquad 0 < C_1 \leq \gamma(x) \leq C_2 \quad \text{on } \overline{\Omega}$$
$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2$$

- Extended to $\gamma \in C^{3/2}(\overline{\Omega})$ (Päivarinta-Panchenko-U, Brown-Torres, 2003)
- $\gamma \in C^{1+\epsilon}(\overline{\Omega}), \gamma$ conormal (Greenleaf-Lassas-U, 2003)
- $\gamma \in C^1(\overline{\Omega})$ (Haberman-Tataru, 2013)
- $\gamma \in W^{1,n}(\overline{\Omega})$, $(n = 3, 4)$ (Haberman, 2015)
- $\gamma \in W^{1,\infty}(\overline{\Omega})$ (Caro-Rogers, 2016)

Conjecture $n \geq 3$, $W^{1,n}(\overline{\Omega})$ is optimal for uniqueness

- Reconstruction A. Nachman (1988)
- Stability G. Alessandrini (1988)
- Numerical Method (Isaacson, Hamilton, Knudsen, Müller, Siltanen, $\cdots$)

DN Map for Schrödinger Equation

Reduction to Schrödinger equation

$$\mathrm{div}(\gamma \nabla w) = 0$$

$$u = \sqrt{\gamma} w$$

Then the equation is transformed into:

$$(\Delta - q)u = 0, q = \frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}}$$

$$\begin{aligned} (\Delta - q)u &= 0 \\ u\big|_{\partial\Omega} &= f \end{aligned}$$

Define $\Lambda_q(f) = \frac{\partial u}{\partial \nu}\big|_{\partial\Omega}$

ν = unit-outer normal to $\partial\Omega$.

Identity

$$\int_\Omega (q_1 - q_2) u_1 u_2 = \int_{\partial\Omega} \left((\Lambda_{q_1} - \Lambda_{q_2}) u_1 \big|_{\partial\Omega} \right) u_2 \big|_{\partial\Omega} dS$$

$$(\Delta - q_i) u_i = 0$$

If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \Lambda_{q_1} = \Lambda_{q_2}$ and

$$\int_\Omega (q_1 - q_2) u_1 u_2 = 0$$

GOAL: Find **MANY** solutions of $(\Delta - q_i) u_i = 0$.

COMPLEX GEOMETRICAL OPTICS

(Sylvester-U) $n \geq 2, \; q \in L^\infty(\Omega)$
Let $\rho \in \mathbb{C}^n$ $(\rho = \eta + ik, \eta, k \in \mathbb{R}^n)$ such that $\boxed{\rho \cdot \rho = 0}$
$(|\eta| = |k|, \eta \cdot k = 0)$.

Then for $|\rho|$ sufficiently large we can find solutions of

$$\boxed{(\Delta - q) w_\rho = 0 \text{ on } \Omega}$$

of the form

$$\boxed{w_\rho = e^{x \cdot \rho}(1 + \Psi_q(x, \rho))}$$

with $\Psi_q \to 0$ in Ω as $|\rho| \to \infty$.

APPLICATIONS

$n \geq 3 \quad (\Delta - q) = 0$, $\boxed{\Lambda_q}$ determines $\boxed{q}$

- EIT $\boxed{\Lambda_\gamma}$ determines $\boxed{\gamma}$
- Optical Tomography (Diffusion Approximation)

$$\boxed{i\omega U - \nabla \cdot D(x) \nabla U + \sigma_a(x) U = 0} \text{ in } \Omega$$

U= Density of photons, D=Diffusion Coefficient, $\sigma_a(x)$= optical absorption.

$\boxed{\text{RESULT}}$

- If $\omega \neq 0$ we can recover both $D(x)$ and $\sigma_a(x)$.
- If $\omega = 0$ we can recover either $D(x)$ or $\sigma_a(x)$.

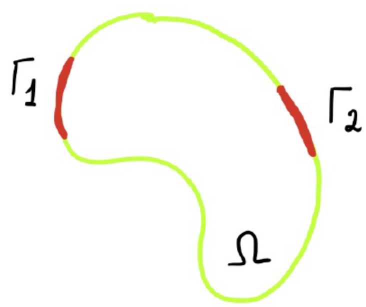
OTHER APPLICATIONS (Fixed energy)

- Optics $(\Delta - k^2 n(x))u = 0$, $n(x)$ isotropic index of refraction $(q(x) = k^2 n(x))$.
- Acoustic $\mathrm{div}(\frac{1}{\rho(x)}\nabla p) + \omega^2 \kappa(x) p = 0$, ρ density, κ compressibility (need two frequencies ω).
- Inverse quantum scattering at fixed energy $(\Delta - q - \lambda^2)u = 0$, q potential.
- Magnetic Schrödinger equation $((-i\nabla + A)^2 + q)u = 0$.
- Maxwell's Equation (Isotropic) (Ola-Somersalo): Reduction to $(\Delta - Q)$, Q an 8×8 matrix.
- Quantitative Photoacoustic Tomography (Bal-U)

Partial Data

Let $\Gamma_1, \Gamma_2 \subset \partial\Omega$ be arbitrary open non-empty. The partial Dirichlet–to–Neumann map,

$$\Lambda_\gamma^{\Gamma_1,\Gamma_2}(f) = (\gamma \partial_\nu u)|_{\Gamma_2}, \quad \text{supp } (f) \subset \Gamma_1.$$



The Calderón problem with partial data: Does $\Lambda_\gamma^{\Gamma_1,\Gamma_2}$ determine γ in Ω? Open in general.

Partial Data

- Bukhgeim–U, 2002:

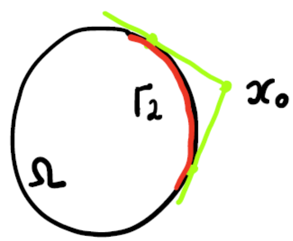
  $\Gamma_1 = \partial\Omega$, $\Gamma_2 = \{x \in \partial\Omega : \xi{\cdot}\nu(x) < \varepsilon\}$, $\xi \in \mathbb{S}^{n-1}$, $\varepsilon > 0$.

  Note: Γ_2 is slightly more than a half of the boundary

- Ammari–U, 2004: $\Gamma_1 = \Gamma_2$, $\gamma_1 = \gamma_2$ near $\partial\Omega$.
- Kenig–Sjöstrand–U, 2007:

$$\Gamma_1 = \text{small neighborhood of complement of} \Gamma_2$$

$$\Gamma_2 = \{x \in \partial\Omega : \frac{(x - x_0)}{|x - x_0|}{\cdot}\nu(x) < \varepsilon\},\ x_0 \notin \overline{ch(\Omega)}, \quad \varepsilon > 0.$$



Partial Data

- Kenig–Salo, 2014: unifies approaches of Kenig–Sjöstrand–U and Isakov and extends both of them.

$\Gamma_1 = \partial\Omega$, Γ_2 as in Kenig–Sjöstrand–U.

Theorem (Krupchyk–U, 2016) Let $\gamma_1, \gamma_2 \in C^{1,\delta}(\overline{\Omega}) \cap H^{\frac{3}{2}}(\Omega)$, $\delta > 0$ arbitrarily small. Assume that $\gamma_1, \gamma_2 > 0$ in $\overline{\Omega}$, $\gamma_1 = \gamma_2$ and $\partial_\nu \gamma_1 = \partial_\nu \gamma_2$ on $\partial\Omega \setminus \Gamma_2$. If $\Lambda_{\gamma_1}^{\Gamma_1,\Gamma_2} = \Lambda_{\gamma_2}^{\Gamma_1,\Gamma_2}$ then $\gamma_1 = \gamma_2$ in Ω.

Remark. Krupchyk–U, 2016: the result holds also for $\gamma_1, \gamma_2 \in W^{1,\infty}(\Omega) \cap H^{\frac{3}{2}+\delta}(\Omega)$, $\delta > 0$.

CGO SOLUTIONS WITH NON-LINEAR PHASE

Kenig-Sjöstrand-U (2007),

$$\boxed{u = e^{\tau(\varphi(x)+i\psi(x))}(a(x) + R(x,\tau))}$$

$\tau \in \mathbb{R}$, φ, ψ real-valued, $R(x,\tau) \to 0$ as $\tau \to \infty$.

φ limiting Carleman weight,

e.g. $\boxed{\varphi(x) = \ln|x - x_0|,}$ $x_0 \notin \overline{ch(\Omega)}$

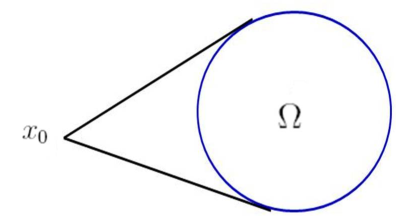
Complex Spherical Waves

$$u = e^{\tau(\varphi(x)+i\psi(x))}a_\tau(x) = e^{\tau(\varphi(x)+i\psi(x))}(a_0(x) + R(x,\tau))$$
$$R(x,\tau) \overset{\tau\to\infty}{\longrightarrow} 0 \text{ in } \Omega$$

$$\varphi(x) = \ln|x - x_0|, \quad x_0 \notin \overline{ch(\Omega)}$$

Eikonal: $\nabla\varphi \cdot \nabla\psi = 0, |\nabla\varphi| = |\nabla\psi|$

$\psi(x) = d(\frac{x-x_0}{|x-x_0|}, \omega), \omega \in S^{n-1}$: smooth for $x \in \bar{\Omega}$.



Transport: $(\nabla\varphi + i\nabla\psi) \cdot \nabla a_\tau = 0$

(Cauchy-Riemann equation in plane generated by $\nabla\varphi, \nabla\psi$)

Carleman Estimates

$u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}|_{\partial\Omega_-} = 0 \qquad \partial\Omega_\pm = \{x \in \partial\Omega; \nabla\varphi\cdot\nu \gtrless 0\}$

$$\int_{\partial\Omega_+} < \nabla\varphi, \nu > |e^{-\tau\varphi(x)}\frac{\partial u}{\partial \nu}|^2 ds \leq \frac{C}{\tau}\int_\Omega |(\Delta - q)ue^{-\tau\varphi(x)}|^2 ds$$

This gives control of $\frac{\partial u}{\partial \nu}|_{\partial\Omega_{+,\delta}}$,

$$\partial\Omega_{+,\delta} = \{x \in \partial\Omega, \nabla\varphi \cdot \nu \geq \delta\}$$

Partial Data

Linearization (Analog of Calderón)

Theorem (Dos Santos Ferreira, Kenig, Sjöstrand-U, 2009; Sjöstrand-U, 2016)

$$\boxed{\int_\Omega huv = 0}$$

$$\Gamma \subseteq \partial\Omega,\ \Gamma \text{ open},$$

$$(\Delta - q)u = (\Delta - q)v = 0, \quad q \text{ is analytic, } u, v \in C^\infty(\overline{\Omega}),$$

$$\text{supp } u|_{\partial\Omega}, \text{ supp } v|_{\partial\Omega} \subseteq \Gamma,$$

$$\Rightarrow \boxed{h = 0}.$$

Complex Spherical Waves

$$\boxed{u_\tau = e^{\tau(\varphi + i\psi)} a_\tau}$$

$$\varphi(x) = \ln|x - x_0|, \; x_0 \notin \overline{\mathrm{ch}(\Omega)}$$

Also used to determine inclusions, obstacles, etc.

a) Conductivity   Ide-Isozaki-Nakata-Siltanen-U, 2007

b) Helmholtz   Nakamura-Yosida, 2007

c) Elasticity   J.-N. Wang-U, 2007

d) Maxwell   T. Zhou, 2010

Complex Spherical Waves

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Anisotropic Case

$\gamma = (\gamma^{ij})$ — positive-definite, symmetric matrix

conductivity

$\Omega \subseteq \mathbb{R}^n, \Omega$ bounded. Under assumptions of no sources or sinks of current the potential u satisfies

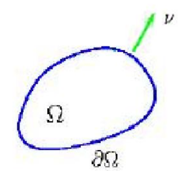
$$\mathrm{div}(\gamma \nabla u) = 0$$

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_i}\right) = 0 \text{ in } \Omega \qquad (*)$$

$$u\big|_{\partial\Omega} = f$$

f = voltage potential at boundary

$$\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \nu^i \gamma^{ij} \frac{\partial u}{\partial x_j}\Big|_{\partial\Omega}$$



Calderón Problem: Can we recover γ in Ω from Λ_γ ?

Invariance

$$\mathrm{div}(\gamma \nabla u) = 0$$
$$u\big|_{\partial\Omega} = f$$

$$\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \gamma^{ij} \nu^i \frac{\partial u}{\partial x_j}\Big|_{\partial\Omega}$$

$$\Lambda_\gamma \Rightarrow \gamma\ ?$$

Answer: No

$$\Lambda_{\psi_*\gamma} = \Lambda_\gamma$$

where $\psi : \Omega \to \Omega$ change of variables

$$\psi|_{\partial\Omega} = \text{Identity}$$

$$\psi_*\gamma = \left(\frac{(D\psi)^T \circ \gamma \circ D\psi}{|\det D\psi|} \right) \circ \psi^{-1}$$

$$v = u \circ \psi^{-1}$$

Open problem: Is this the only obstruction ($\gamma \in C^\infty(\bar{\Omega})$)?

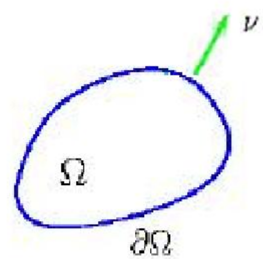
Geometric Inverse Problem (Lee-U, 1989)

(M, g) compact Riemannian manifold with boundary.

$n \geq 3$, Δ_g Laplace-Beltrami operator $g = (g_{ij})$ pos. def. symmetric matrix

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial u}{\partial x_j}\right) \qquad (g^{ij}) = (g_{ij})^{-1}$$

$$\Delta_g u = 0 \text{ on } M$$
$$u\big|_{\partial M} = f$$

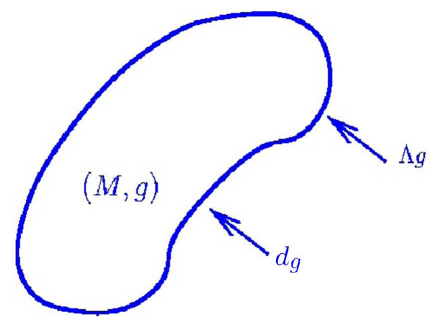


Conductivity:
$\gamma^{ij} = \sqrt{\det g}\, g^{ij}$

$$\Lambda_g(f) = \sum_{i,j=1}^{n} \nu^j g^{ij} \frac{\partial u}{\partial x_i} \sqrt{\det g}\,\Big|_{\partial M}$$

ANOTHER MOTIVATION (STRING THEORY)

HOLOGRAPHY



Dirichlet-to-Neumann map is the "boundary-2pt function"

Inverse problem: Can we recover (M, g) (bulk) from boundary-2pt function ?

M. Parrati and R. Rabadan, Boundary rigidity and holography, JHEP 0401 (2004) 034

Anisotropic ($n \geq 3$)

Theorem ($n \geq 3$) (Lassas-U 2001, Lassas-Taylor-U 2003) $(M, g_i), i = 1, 2$, real-analytic, connected, compact, Riemannian manifolds with boundary. Let $\Gamma \subseteq \partial M$, Γ open. Assume

$$\boxed{\Lambda_{g_1}(f)|_\Gamma = \Lambda_{g_2}(f)|_\Gamma,} \quad \forall f,\ f \text{ supported in } \Gamma$$

Then $\exists \psi : M \to M$ diffeomorphism, $\psi\big|_\Gamma =$ Identity, so that

$$\boxed{g_1 = \psi^* g_2}$$

In fact one can determine topology of M, as well (only need to know $\Lambda_g, \partial M$).

Non-Uniqueness (Anisotropic, $n \geq 3$)

Counterexamples (Daudé-Kamran-Nicoleau, 2019)

- $\Gamma_1 = \Gamma_2$, γ is smooth in the interior but only Hölder up to the boundary (based on Müller's counterexamples for unique continuation).

- Γ_1 and Γ_2 are disjoint, γ is smooth up to the boundary.

Moding Out the Diffeomorphism Group

Some conformal class $\boxed{\Lambda_{\beta g} = \Lambda_g}, \quad \beta \in C^\infty(M)$

$$\Longrightarrow \beta = 1?$$

More general problem

$$\boxed{\begin{array}{c} (\Delta_g - q)u = 0, \;\; q \in C^\infty(M) \\ u|_{\partial M} = f, \\ \Lambda_g(f) = \frac{\partial u}{\partial \nu_g}|_{\partial M}. \end{array}}$$

Inverse Problem: Does $\boxed{\Lambda_g}$ determines $\boxed{q}$?

Moding Out the Diffeomorphism Group
$(n \geq 3)$

$$
\begin{aligned}
(\Delta_g - q)u = 0,\ & q \in C^\infty(M) \\
u|_{\partial M} &= f, \\
\Lambda_g(f) &= \frac{\partial u}{\partial \nu_g}|_{\partial M}.
\end{aligned}
$$

$$(*) \quad g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}, \quad c > 0.$$

Theorem (Dos Santos-Kenig-Salo-U, 2009) Assume that there is a global coordinate system so that (*) is true. In addition g_0 is simple. Then Λ_g determines uniquely q.

Extension: Dos Santos-Kurylev-Lassas-Salo, 2016

Moding Out the Diffeomorphism Group

$$g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}, \quad x' \in \mathbb{R}^{n-1}$$

Examples

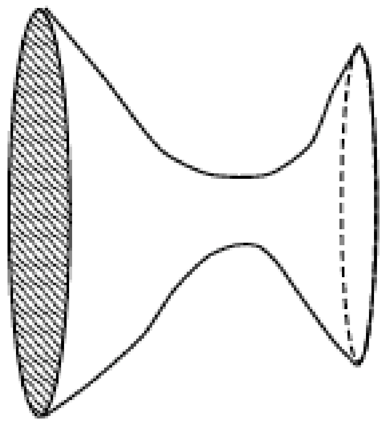
(a) $g(x)$ conformal to Euclidean metric (Sylvester-U, 1987)

(b) $g(x)$ conformal to hyperbolic metric (Isozaki, 2004)

(c) $g(x)$ conformal to metric on sphere (minus a point)

Non-uniqueness for EIT (Cloaking)

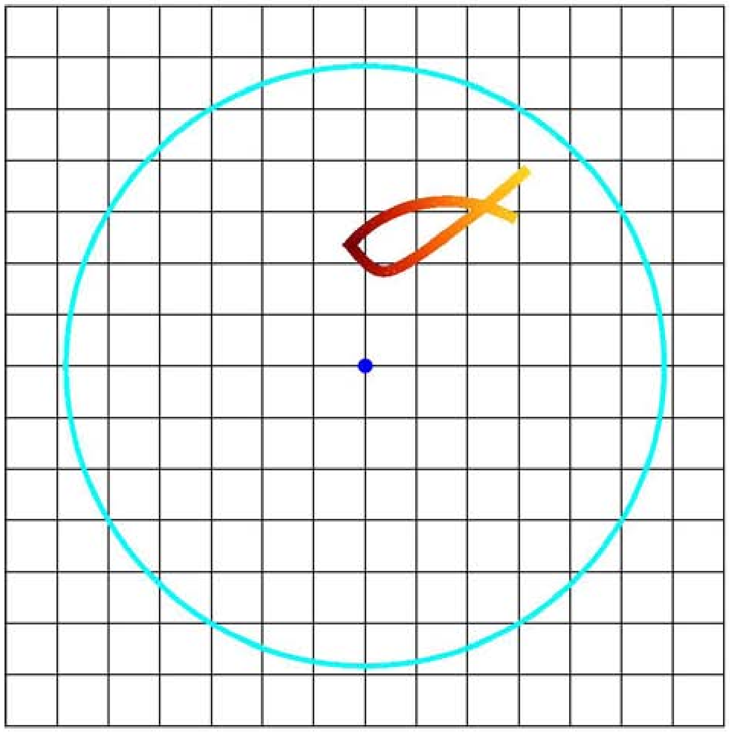
Motivation (Greenleaf-Lassas-U, MRL, 2003)



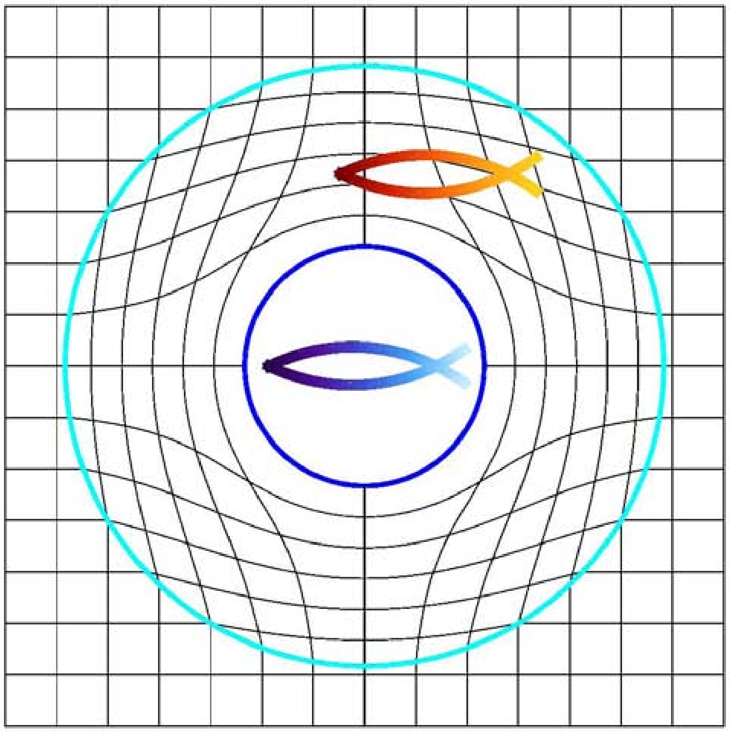
When bridge connecting the two parts of the manifold gets narrower the boundary measurements give less information about isolated area.

When we realize the manifold in Euclidean space we should obtain conductivities whose boundary measurements give no information about certain parts of the domain.

Transformation Optics



virtual space



physical space

Cloaking

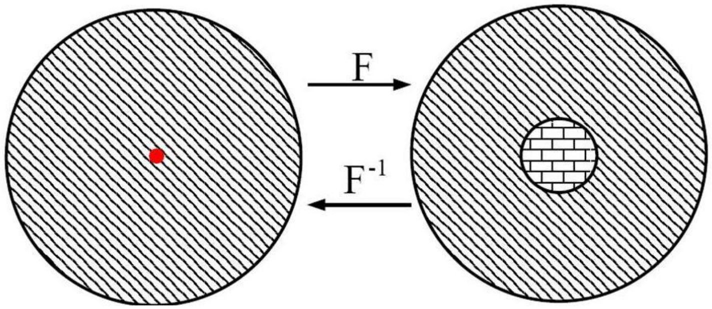
Greenleaf-Lassas-U, 2003

Let $\Omega = \mathcal{B}(0,2) \subset \mathbb{R}^3$, $D = \mathcal{B}(0,1)$, where $\mathcal{B}(0,r) = \{x \in \mathbb{R}^3;\ |x| < r\}$

$$F: \Omega \setminus \{0\} \to \Omega \setminus \overline{D}$$

$$F(x) = \left(\frac{|x|}{2} + 1\right)\frac{x}{|x|}$$

F - diffeomorphism, $\quad F|_{\partial\Omega} = \text{Identity}$



Cloaking

Let $\quad \begin{array}{l} \gamma = g = \text{identity on } \mathcal{B}(0,2), \\ \widehat{\gamma} = F_*\gamma \text{ on } \mathcal{B}(0,2)\backslash\mathcal{B}(0,1), \\ \widehat{g} = \text{metric associated to } \widehat{\gamma}. \end{array}$

In spherical coordinates

$$(r,\phi,\theta) \to (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta),$$

$$\widehat{\gamma} = \begin{pmatrix} 2(r-1)^2\sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 \\ 0 & 0 & 2(\sin\theta)^{-1} \end{pmatrix}$$

Let $\widetilde{\gamma}$ (*resp.* $\widetilde{g}$) be the conductivity (*resp.* metric) in $\mathcal{B}(0,2)$ such that $\widetilde{\gamma} = \widehat{\gamma}$ (*resp.* $\widetilde{g} = \widehat{g}$) on $\mathcal{B}(0,2)\backslash\mathcal{B}(0,1)$ and arbitrarily positive definite on $B(0,1)$. Then

Theorem (Greenleaf-Lassas-U 2003)

$$\boxed{\Lambda_{\widetilde{\gamma}} = \Lambda_{\gamma}} \quad \left(\textit{resp. } \boxed{\Lambda_{\widetilde{g}} = \Lambda_{g}}\right)$$

The Two Dimensional Case

Theorem ($n = 2$) Let $\gamma_j \in C^2(\overline{\Omega})$, $j = 1, 2$.

Assume $\boxed{\Lambda_{\gamma_1} = \Lambda_{\gamma_2}}$. Then $\boxed{\gamma_1 = \gamma_2}$.

- Nachman (1996)
- Brown-U (1997) Improved to γ_j Lipschitz
- Astala-Päivärinta (2006) Improved to $\gamma_j \in L^\infty(\Omega)$

Schrödinger (n=2)

This follows from more general result

Theorem ($n = 2$, Bukhgeim, 2008) Let $q_j \in L^\infty(\Omega)$, $j = 1, 2$.

Assume $\boxed{\Lambda_{q_1} = \Lambda_{q_2}}$. Then $\boxed{q_1 = q_2}$.

Bukhgeim, $q_j \in C^1(\bar{\Omega})$

Extension: Blåsten, Imanuvilov, Yamamoto, 2016

CGO (n=2)

$$\boxed{\Lambda_{q_1} = \Lambda_{q_2}} \Rightarrow \boxed{q_1 = q_2}$$

Sketch of proof New class of CGO solutions

$$\begin{aligned} u_1(z,\tau) &= e^{\tau z^2}(1 + r_1(z,\tau)) \\ u_2(z,\tau) &= e^{-\tau \bar{z}^2}(1 + r_2(z,\tau)) \end{aligned} \qquad \tau \gg 1$$

solve $(\Delta - q_j)u_j = 0$ with $r_j(z,\tau) \to 0$ on Ω sufficiently fast.

<u>**Notation**</u> $z = x_1 + ix_2$

Remark $z^2 = x_1^2 - x_2^2 + 2ix_1x_2 = \varphi + i\psi$

$$\nabla\varphi \cdot \nabla\psi = 0, \quad |\nabla\varphi| = |\nabla\psi|$$

φ harmonic, ψ conjugate harmonic.

General result: $\Delta_g - q$, M Riemann surface (Guillarmou-Tzou , 2010)

Partial Data for Second Order Elliptic Equations (n=2)

(Imanuvilov–U–Yamamoto, 2011)

$$\boxed{\Delta_g + A(z)\frac{\partial}{\partial z} + B(z)\frac{\partial}{\partial \overline{z}} + q} \qquad z = x_1 + ix_2$$

$g = (g_{ij})$ positive definite symmetric matrix;

$$\Delta_g u = \frac{1}{\sqrt{\det(g)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det(g)}\, g^{ij} \frac{\partial u}{\partial x_j}\right) \qquad g^{ij} = (g_{ij})^{-1}$$

Includes:

- Anisotropic Calderón's Problem
- Magnetic Schrödinger Equation
- Convection terms

Anisotropic (n=2)

Theorem (Imanuvilov–U–Yamamoto, 2011) $\Omega \subset \mathbb{R}^2$, $\Gamma \subset \partial\Omega$, Γ open, $\gamma_k = (\gamma_k^{ij}) \in C^\infty(\bar{\Omega}), k = 1, 2$, positive definite symmetric. Assume

$$\Lambda_{\gamma_1}(f)|_\Gamma = \Lambda_{\gamma_2}(f)|_\Gamma, \quad \forall f \quad \text{supp} f \subset \Gamma.$$

Then $\exists F : \bar{\Omega} \to \bar{\Omega}$, C^∞ diffeomorphism, $F|_\Gamma =$ Identity such that

$$F_*\gamma_1 = \gamma_2.$$

Full Data ($\Gamma = \partial\Omega$):

- $\gamma_k \in C^2(\bar{\Omega})$, Nachman (1996)
- γ_k Lipschitz, Sun–U (2001)
- $\gamma_k \in L^\infty(\Omega)$, Astala–Lassas–Päivärinta (2006)

Fractional Laplacian

Consider the *fractional Laplacian*

$$(-\Delta)^s, \quad 0 < s < 1,$$

defined via the Fourier transform by

$$(-\Delta)^s u = \mathcal{F}^{-1}\{|\xi|^{2s}\,\widehat{u}(\xi)\}.$$

This operator is *nonlocal*: it does not preserve supports, and computing $(-\Delta)^s u(x)$ involves values of u far away from x.

Fractional Laplacian

Different models for diffusion:

$\partial_t u - \Delta u = 0$	normal diffusion/BM
$\partial_t u + (-\Delta)^s u = 0$	superdiffusion/Lévy flight
$\partial_t^\alpha u - \Delta u = 0$	subdiffusion/CTRW

The *fractional Laplacian* is related to

- anomalous diffusion involving long range interactions (turbulent media, population dynamics)
- Lévy processes in probability theory
- financial modelling with jump processes

Many results for time-fractional inverse problems, very few for space-fractional [Jin-Rundell, 2015].

Fractional Laplacian

Let $\Omega \subset \mathbb{R}^n$ bounded, $q \in L^\infty(\Omega)$. Since $(-\Delta)^s$ is nonlocal, the Dirichlet problem becomes

$$\begin{cases} ((-\Delta)^s + q)u = 0 & \text{in } \Omega, \\ u = f & \text{in } \Omega_e \end{cases}$$

where $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ is the *exterior domain*.

Given $f \in H^s(\Omega_e)$, look for a solution $u \in H^s(\mathbb{R}^n)$. DN map

$$\Lambda_q : H^s(\Omega_e) \to H^{-s}(\Omega_e), \quad \Lambda_q f = (-\Delta)^s u|_{\Omega_e}.$$

Inverse problem: given Λ_q, determine q.

First Result

Theorem(Ghosh–Salo–U, 2020)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $0 < s < 1$, and let $q_1, q_2 \in L^\infty(\Omega)$. If $W, W' \subset \Omega_e$ are open sets, and if

$$\Lambda_{q_1} f|_{W'} = \Lambda_{q_2} f|_{W'}, \quad f \in C_c^\infty(W),$$

then $q_1 = q_2$ in Ω.

Main features:

- local data result for *arbitrary* $W, W' \subset \Omega_e$
- the same method works for *all* $n \geq 2$
- new mechanism for solving (nonlocal) inverse problems

Main tools 1: uniqueness

Theorem
If $u \in H^{-r}(\mathbb{R}^n)$ for some $r \in \mathbb{R}$, and if $u|_W = (-\Delta)^s u|_W = 0$ for some open set $W \subset \mathbb{R}^n$, then $u \equiv 0$.

Proof (sketch). If u is nice enough, then

$$(-\Delta)^s u \sim \lim_{y \to 0} y^{1-2s} \partial_y w(\,\cdot\,, y)$$

where $w(x, y)$ is the *Caffarelli-Silvestre extension* of u:

$$\begin{cases} \operatorname{div}_{x,y}(y^{1-2s} \nabla_{x,y} w) = 0 & \text{in } \mathbb{R}^n \times \{y > 0\}, \\ w|_{y=0} = u. & \end{cases}$$

Thus $(-\Delta)^s u$ is obtained from a *local equation*, which is degenerate elliptic with A_2 weight y^{1-2s}. Carleman estimates [Rüland 2015] and $u|_W = (-\Delta)^s u|_W = 0$ imply uniqueness.

Main tools 2: approximation

Theorem(Ghosh–Salo–U, 2020)
Any $f \in L^2(\Omega)$ can be approximated in $L^2(\Omega)$ by *solutions* $u|_\Omega$, where

$$((-\Delta)^s + q)u = 0 \text{ in } \Omega, \qquad \text{supp}(u) \subset \overline{\Omega} \cup \overline{W}. \quad (*)$$

If everything is C^∞, any $f \in C^k(\overline{\Omega})$ can be approximated in $C^k(\overline{\Omega})$ by functions $d(x)^{-s}u|_\Omega$ with u as in $(*)$.

Proof. Apply this to

$$\int_\Omega (q_1 - q_2)u_1u_2 = 0, \quad (-\Delta)^s + q_j)u_j = 0 \, (j = 1, 2)$$

Further Results

- Variable coefficients fractional elliptic operators (Ghosh–Lin–Xiao, 2017)
- Regularity and stability (Rüland–Salo, 2017)
- Reconstruction and single Measurement (Ghosh–Rüland–Salo–U, 2018)
- Fractional Schrödinger equation with drift (Cekić–Lin–Rüland, 2018)
- Non-local Perturbations (Bhattacharyya–ghosh–U, 2019)
- Fractional magnetic operators (Covi, 2019, Li, 2020)
- Any local perturbation of fractional Laplacian (Covi–Mönkkönen–Railo–U, 2020)