

Let $\Omega = \mathbb{R}_+^n = \{x_n > 0\}$, so $\partial\Omega = \mathbb{R}^{n-1} = \{x_n = 0\}$.

Want to compute DN map for Laplace equation in Ω . Consider

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}_+^n \\ u|_{x_n=0} = f \end{cases}$$

Writing $x = (x', x_n)$ and taking Fourier transforms in x' gives

$$\begin{cases} (\partial_n^2 - |z'|^2) \hat{u}(z', x_n) = 0 \\ \hat{u}(z', 0) = \hat{f}(z') \end{cases}$$

Solving this ODE for fixed z' and choosing the solution that decays in z' (for $x_n > 0$) gives

$$\hat{u}(z', x_n) = e^{-x_n |z'|} \hat{f}(z')$$

$$\Rightarrow u(x', x_n) = \mathcal{F}_{z'}^{-1} \{ e^{-x_n |z'|} \hat{f}(z') \}$$

We may now compute the DN map:

$$\Delta_\Omega f = -\partial_n u|_{x_n=0} = \mathcal{F}_{z'}^{-1} \{ |z'| \hat{f}(z') \}$$

Thus the DN map on the boundary $\mathbb{R}^{n-1}$ of $\mathbb{R}_+^n$ is just the Fourier multiplier $|z'|$. This is an elliptic Ψ DO of order 1 (if $\psi \in C_c^\infty(\mathbb{R}^{n-1})$ satisfies $\psi = 1$ near 0, one can write

$$|z'| = \underbrace{(1 - \psi(z')) |z'|}_{\text{smooth Kohn-Nirenberg symbol of order 1}} + \underbrace{\psi(z') |z'|}_{\text{smoothing symbol (compactly supported in } z')}$$