$P = -\Delta,\qquad \hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\,dx$

$-\Delta u = f$ in $\mathbb{R}^n \iff |\xi|^2 \hat{u}(\xi) = \hat{f}(\xi)$ for $\xi \in \mathbb{R}^n$

$\iff \hat{u}(\xi) = \frac{1}{|\xi|^2}\hat{f}(\xi) \iff u = \mathcal{F}^{-1}\left\{\frac{1}{|\xi|^2}\hat{f}(\xi)\right\}$

Similarly for $A = \sum_{|\alpha|\le m} a_\alpha D^\alpha$, $a_\alpha \in \mathbb{C}$.

Variable coeffs: $A(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$, $a_\alpha \in C^\infty(\mathbb{R}^n)$, $\partial^\beta a_\alpha \in L^\infty(\mathbb{R}^n)$ $\forall \alpha,\beta$

Compute $A u(x) = A\left[(2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{u}(\xi)\,d\xi\right]$

$$= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\Big(\underbrace{\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha}_{a(x,\xi)}\Big)\hat{u}(\xi)\,d\xi$$

Here $a(x,\xi) := \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ is the (full) symbol of $A(x,D)$

Try to define an operator

$$Bf(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\frac{1}{a(x,\xi)}\hat{f}(\xi)\,d\xi$$

Microlocal analysis: $u = Bf$ is a well-defined approximate solution of $Au = f$ in $\mathbb{R}^n$ if A is elliptic (its principal symbol $a_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ is nonvanishing for $\xi \neq 0$).

In general, a ΨDO is an operator

$$Af(x) = (2\pi)^{-n}\int e^{ix\cdot\xi} a(x,\xi)\hat{f}(\xi)\,d\xi$$

where $a(x,\xi)$ belongs to some symbol class S^m (typically contains polynomials in ξ and inverses). One writes $A = \mathrm{Op}(a)$ and $A \in \Psi^m$. Typical statements:

- $A \in \Psi^m$, $B \in \Psi^{m'}$ $\Rightarrow$ $AB \in \Psi^{m+m'}$ has principal symbol $\sigma_{pr}(AB) = \sigma_{pr}(A)\sigma_{pr}(B)$
- $A \in \Psi^m$ $\Rightarrow$ A bounded $H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$ $\forall s \in \mathbb{R}$
- $A \in \Psi^m$ elliptic $\Rightarrow$ $\exists B \in \Psi^{-m}$ with $AB = I + R$, $R \in \Psi^{-\infty}$

Microlocal analysis is the theory of ΨDOs and FIOs. Many settings:
started in 1950s

- in $\mathbb{R}^n$
- in open sets of $\mathbb{R}^n$
- on compact manifolds without boundary, possibly on vector bundles
- " " with boundary (Boutet de Monvel calculus)
- on non-compact manifolds (scattering calculus)
- with asymptotic parameters (semiclassical calculus)

Applications:

1) Calderon problem in $\Omega \subset \mathbb{R}^n$ bounded C^∞ domain:
$$\begin{cases} \operatorname{div}(\gamma(x)\nabla u) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$
DN map $\Lambda_\gamma : f \mapsto \partial_\nu u|_{\partial\Omega}$ is in $\Psi^1(\partial\Omega)$.
Principal symbol of Λ_γ det. $\gamma|_{\partial\Omega}$, sub-principal symbol det. $\partial_\nu\gamma|_{\partial\Omega}$, ..., and full symbol of Λ_γ (in certain coordinates) det. Taylor series of γ on $\partial\Omega$.

2) Geodesic X-ray transform in compact mfld (M,g) with boundary: if $f \in C^\infty(M)$
$$If(\gamma) = \int_\gamma f(\gamma(t))\,dt.$$

Under certain assumptions on (M,g),
I is a FIO of order $-1/2$ and I^*I is an elliptic ΨDO of order -1 on M. Microlocal analysis:
if $If = 0$, then $f \in C^\infty(M)$ and thus main singularities of f can be recovered from If.

3) Gauss–Bonnet theorem: if (M,g) is a closed oriented 2D mfld, then
$$\frac{1}{2\pi}\int_M K\,dV = \chi(M)$$
(special case of Atiyah–Singer index theorem)

Here $\chi(M)$ is the Fredholm index of $A = d + d^* : \Lambda^{\text{even}} \to \Lambda^{\text{odd}}$
$(\operatorname{Ind}(A) = \dim \operatorname{Ker}(A) - \dim \operatorname{Ker}(A^*) = b_0(M) - b_1(M) + b_2(M))$,
and $\frac{1}{2\pi}\int_M K\,dV$ is $\operatorname{ind}(A)$ computed in another way.

4) Eigenvalue asymptotics: let $\Omega \subset \mathbb{R}^n$ be a bounded domain, let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \to \infty$ be Dirichlet ews of $-\Delta$ in Ω:
$$\begin{cases} -\Delta\phi_j = \lambda_j\phi_j & \text{in } \Omega \\ \phi_j|_{\partial\Omega} = 0, \end{cases}$$
and let $N(\lambda) = \#\{\lambda_j \,;\, \lambda_j \leq \lambda\}$. Then
$$N(\lambda) \sim \frac{|\Omega|}{\Gamma(\frac{n+2}{2})(4\pi)^{n/2}}\lambda^{n/2} \quad \text{as } \lambda \to \infty.$$
Proof: estimate heat kernel $\operatorname{Tr}(e^{-t\Delta}) = \sum_{j=0}^{\infty} e^{-t\lambda_j}$ as $t \to 0$ by microlocal methods, and use a Tauberian theorem.