Calderón problem for the fractional Laplacian

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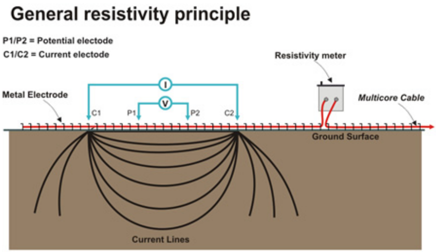
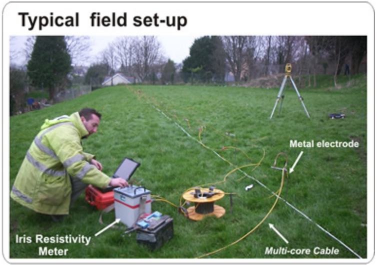
Outline

1. Calderón problem
2. Fractional Laplacian
3. Approximation property

Calderón problem

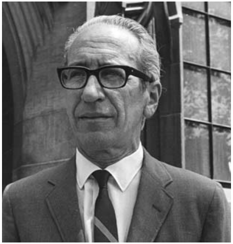
Electrical Resistivity Imaging in geophysics (1920's) [image: TerraDat]





A.P. Calderón (1980):

- mathematical formulation
- solution of the linearized problem
- exponential solutions



Calderón problem

Conductivity equation

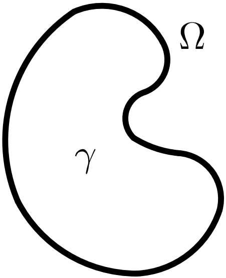
$$\begin{cases} \operatorname{div}(\gamma(x)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ bounded Lipschitz domain, $\gamma \in L^\infty(\Omega)$ positive scalar function (electrical conductivity).

Boundary measurements given by *Dirichlet-to-Neumann (DN) map*[1]

$$\Lambda_\gamma : f \mapsto \gamma \nabla u \cdot \nu|_{\partial\Omega}.$$

Inverse problem: given Λ_γ, determine γ.



[1] as a map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$

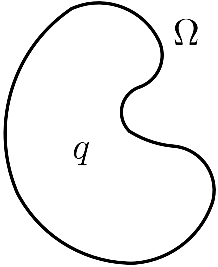
Schrödinger equation

Substitute $u = \gamma^{-1/2} v$, conductivity equation $\mathrm{div}(\gamma \nabla u) = 0$ reduces to Schrödinger equation $(-\Delta + q)v = 0$ where

$$q = \frac{\Delta(\gamma^{1/2})}{\gamma^{1/2}}.$$

If $q \in L^\infty(\Omega)$, consider Dirichlet problem

$$\begin{cases} (-\Delta + q)u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$



The DN map is $\Lambda_q : f \mapsto \partial_\nu u|_{\partial\Omega}$.

Inverse problem: given Λ_q, determine q.

Calderón problem

Model case of inverse boundary problems for elliptic equations (Schrödinger, Maxwell, elasticity).

Related to:

- optical and hybrid imaging methods
- inverse scattering
- geometric problems (boundary rigidity)
- invisibility cloaking

Calderón problem

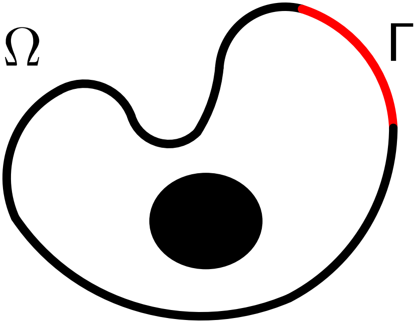
Uniqueness results:

$n \geq 3$	$q \in L^\infty$	Sylvester-Uhlmann 1987
	$q \in L^{n/2}$	Chanillo/Jerison-Kenig/Lavine-Nachman 1990
$n = 2$	$q \in C^1$	Bukhgeim 2008
	$q \in L^{2+\varepsilon}$	Blåsten-Imanuvilov-Yamamoto 2015

Connections to *Carleman estimates* and *unique continuation* (u vanishes in a ball $\implies u \equiv 0$).

Local data problem

Prescribe voltages on Γ, measure currents on Γ:



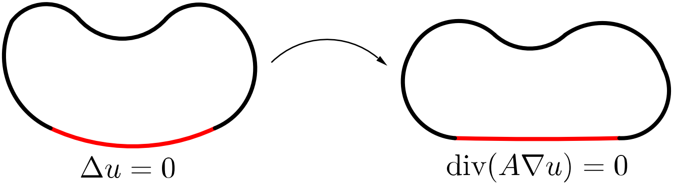
Measure $\Lambda_\gamma f|_\Gamma$ for any f with $\mathrm{supp}(f) \subset \Gamma$.

Local data problem

Uniqueness known

- if $n = 2$ for any $\Gamma \subset \partial\Omega$ [Imanuvilov-Uhlmann-Yamamoto 2010]
- if $n \geq 3$ and inaccessible part has a conformal symmetry (e.g. flat, cylindrical or part of a surface of revolution) [Kenig-S 2013, Isakov 2007, Kenig-Sjöstrand-Uhlmann 2007]

Issue: flattening the boundary results in *matrix conductivities*.



Calderón problem for $\mathrm{div}(A\nabla u) = 0$, $A = (a^{jk})$, open if $n \geq 3$!

Outline

Fractional Laplacian

We will study an inverse problem for the *fractional Laplacian*

$$(-\Delta)^s, \quad 0 < s < 1,$$

defined via the Fourier transform by

$$(-\Delta)^s u = \mathscr{F}^{-1}\{|\xi|^{2s}\hat{u}(\xi)\}.$$

This operator is *nonlocal*: it does not preserve supports, and computing $(-\Delta)^s u(x)$ involves values of u far away from x.

Fractional Laplacian

Different models for diffusion:

$\partial_t u - \Delta u = 0$	normal diffusion/BM
$\partial_t u + (-\Delta)^s u = 0$	superdiffusion/Lévy flight
$\partial_t^\alpha u - \Delta u = 0$	subdiffusion/CTRW

The *fractional Laplacian* is related to

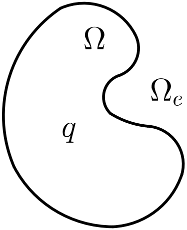
- anomalous diffusion involving long range interactions (turbulent media, population dynamics)
- Lévy processes in probability theory
- financial modelling with jump processes

Many results for time-fractional inverse problems, very few for space-fractional [Jin-Rundell, tutorial 2015].

Fractional Laplacian

Let $\Omega \subset \mathbb{R}^n$ bounded, $q \in L^\infty(\Omega)$. Since $(-\Delta)^s$ is nonlocal, the Dirichlet problem becomes

$$\begin{cases} ((-\Delta)^s + q)u = 0 & \text{in } \Omega, \\ u = f & \text{in } \Omega_e \end{cases}$$



where $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ is the *exterior domain*.

Given $f \in H^s(\Omega_e)$, look for a solution $u \in H^s(\mathbb{R}^n)$. DN map

$$\Lambda_q : H^s(\Omega_e) \to H^{-s}(\Omega_e), \quad \Lambda_q f = (-\Delta)^s u|_{\Omega_e}.^1$$

Inverse problem: given Λ_q, determine q.

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[1] the work required to maintain Dirichlet data f in Ω_e

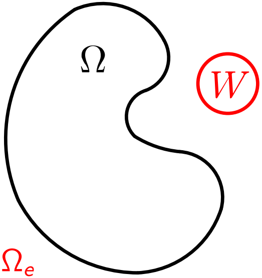
Main result

Theorem (Ghosh-S-Uhlmann 2016)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $0 < s < 1$, and let $q_1, q_2 \in L^\infty(\Omega)$. If $W \subset \Omega_e$ is any open set, and if

$$\Lambda_{q_1} f|_W = \Lambda_{q_2} f|_W, \quad f \in C_c^\infty(W),$$

then $q_1 = q_2$ in Ω.



Main features:

- local data result for *arbitrary* $W \subset \Omega_e$
- the same method works for *all* $n \geq 2$
- new mechanism for solving (nonlocal) inverse problems

Outline

1. Calderón problem
2. Fractional Laplacian
3. Approximation property

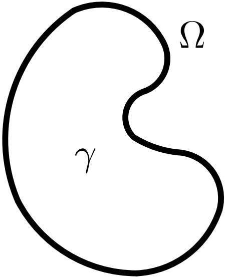
Calderón problem

Recall the standard Calderón problem:

$$\begin{cases} \operatorname{div}(\gamma(x)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

Boundary measurements given by *DN map*

$$\Lambda_\gamma : f \mapsto \gamma \nabla u \cdot \nu|_{\partial\Omega}.$$



Inverse problem: given Λ_γ, determine γ.

Two methods:

1. Runge approximation property
2. Complex geometrical optics solutions

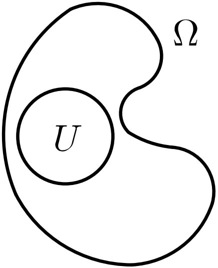
Runge approximation

Classical Runge property (for $\overline{\partial} u = 0$):

analytic functions in simply connected $U \subset \mathbb{C}$ can be approximated by complex polynomials.

General Runge property (for elliptic PDE):

any solution in U, where $U \subset \Omega \subset \mathbb{R}^n$, can be approximated using solutions in Ω.



Reduces by duality to the *unique continuation principle* [Lax, Malgrange 1956], cf. approximate controllability.

Runge approximation

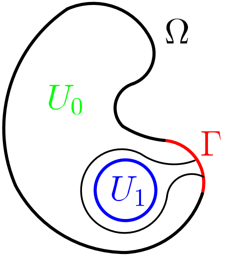
Theorem (cf. localized potentials, Harrach 2008)

Let $\Gamma \subset \partial\Omega$ open, $\gamma \in W^{1,\infty}(\Omega)$ positive. If $U_0, U_1 \subset \Omega$ are open sets so that

$$\overline{U}_0 \cap \overline{U}_1 = \emptyset, \quad \Omega \setminus (\overline{U}_0 \cup \overline{U}_1) \text{ meets } \Gamma,$$

then $\exists\ u_j \in H^1(\Omega)$, $\mathrm{div}(\gamma \nabla u_j) = 0$, with

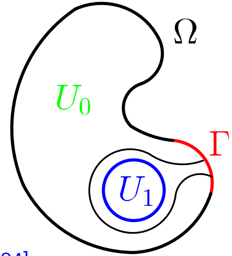
$$u_j|_{U_0} \approx 0, \quad u_j|_{U_1} \approx j,$$
$$\mathrm{supp}(u_j|_{\partial\Omega}) \subset \Gamma.$$



Proof. Apply Runge approximation to piecewise constant solutions $w_j \in H^1(U_0 \cup U_1)$ with $w_j|_{U_0} = 0$, $w_j|_{U_1} = j$. □

Runge approximation

Produce solutions with $u|_{U_0} \approx 0$ and $u|_{U_1} \gg 1$ (region of interest), but with *very little control outside $U_0 \cup U_1$*. Useful in the Calderón problem for



- boundary determination [Kohn-Vogelius 1984]
- piecewise analytic conductivities [Kohn-Vogelius 1985]
- local data if γ is known near $\partial\Omega$ [Ammari-Uhlmann 2004]
- detecting shapes of obstacles (γ known near $\partial\Omega$), e.g.
  - singular solutions [Isakov 1988]
  - probe method [Ikehata 1998]
  - oscillating-decaying solutions [Nakamura-Uhlmann-Wang 2005]
  - monotonicity tests [Harrach 2008]

Complex geometrical optics

Runge type results use that γ *is known near $\partial\Omega$*, or employ *monotonicity conditions*. They do not allow to determine conductivities in $C^\infty(\overline{\Omega})$, which may oscillate near $\partial\Omega$.

Complex geometrical optics solutions [Sylvester-Uhlmann 1987]

$$u = e^{\rho\cdot x}(1+r), \qquad \rho \in \mathbb{C}^n, \qquad \rho\cdot\rho = 0$$

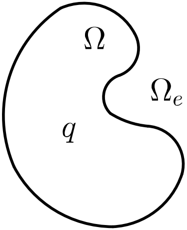
where $\|r\|_{L^2(\Omega)} \to 0$ as $|\rho| \to \infty$.

These solutions are small in $\{\mathrm{Re}(\rho)\cdot x < 0\}$, large in $\{\mathrm{Re}(\rho)\cdot x > 0\}$, and oscillate in the direction of $\mathrm{Im}(\rho)$. Unlike in Runge approximation, solutions are *controlled in all of Ω*, and yield the Fourier transform of the unknown coefficient.

Fractional problem

Return to fractional problem: $\Omega \subset \mathbb{R}^n$ bounded, $q \in L^\infty(\Omega)$,

$$\begin{cases} ((-\Delta)^s + q)u = 0 & \text{in } \Omega, \\ u = f & \text{in } \Omega_e. \end{cases}$$



DN map

$$\Lambda_q : f \mapsto (-\Delta)^s u|_{\Omega_e}.$$

Theorem (Ghosh-S-Uhlmann 2016)

Let $0 < s < 1$, and let $q_1, q_2 \in L^\infty(\Omega)$. If $W \subset \Omega_e$ is any open set, and if

$$\Lambda_{q_1} f|_W = \Lambda_{q_2} f|_W, \qquad f \in C_c^\infty(W),$$

then $q_1 = q_2$ in Ω.

Main tools 1: uniqueness

The fractional equation has strong uniqueness properties:

Theorem

If $u \in H^{-r}(\mathbb{R}^n)$ for some $r \in \mathbb{R}$, and if both u and $(-\Delta)^s u$ vanish in some open set, then $u \equiv 0$.

Essentially due to [M. Riesz 1938], also have strong unique continuation results [Fall-Felli 2014, Rüland 2015].

Such a result could never hold for the Laplacian:
if $u \in C_c^\infty(\mathbb{R}^n)$, then both u and Δu vanish in a large set.

Main tools 1: uniqueness

Theorem

If $u \in H^{-r}(\mathbb{R}^n)$ for some $r \in \mathbb{R}$, and if $u|_W = (-\Delta)^s u|_W = 0$ for some open set $W \subset \mathbb{R}^n$, then $u \equiv 0$.

Proof (sketch). If u is nice enough, then

$$(-\Delta)^s u \sim \lim_{y \to 0} y^{1-2s} \partial_y w(\,\cdot\,, y)$$

where $w(x, y)$ is the *Caffarelli-Silvestre extension* of u:

$$\begin{cases} \mathrm{div}_{x,y}(y^{1-2s} \nabla_{x,y} w) = 0 & \text{in } \mathbb{R}^n \times \{y > 0\}, \\ w\big|_{y=0} = u. \end{cases}$$

Thus $(-\Delta)^s u$ is obtained from a *local equation*, which is degenerate elliptic with A_2 weight y^{1-2s}. Carleman estimates [Rüland 2015] and $u|_W = (-\Delta)^s u|_W = 0$ imply uniqueness. □

Main tools 2: approximation

The fractional equation has strong approximation properties (approximate control in all of Ω, this could never hold for Δ):

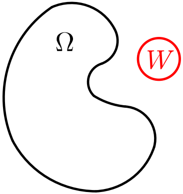
Theorem (Ghosh-S-Uhlmann 2016)

Any $f \in L^2(\Omega)$ can be approximated in $L^2(\Omega)$ by *solutions $u|_\Omega$*, where

$$((-\Delta)^s + q)u = 0 \text{ in } \Omega, \qquad \mathrm{supp}(u) \subset \overline{\Omega} \cup \overline{W}. \qquad (*)$$

If everything is C^∞, any $f \in C^k(\overline{\Omega})$ can be approximated in $C^k(\overline{\Omega})$ by functions $d(x)^{-s}u|_\Omega$ with u as in $(*)$.

Earlier [Dipierro-Savin-Valdinoci 2016]: any $f \in C^k(\overline{B}_1)$ can be approximated in $C^k(\overline{B}_1)$ by solutions of $(-\Delta)^s u = 0$ in B_1, but with very little control over $\mathrm{supp}(u)$.



Main tools 2: approximation

The approximation property follows by duality from the uniqueness result.

This uses Fredholm properties of the solution operator for

$$\begin{cases} ((-\Delta)^s + q)u = F & \text{in } \Omega, \\ u = 0 & \text{in } \Omega_e, \end{cases}$$

mapping $F \in H^{\alpha-2s}(\Omega)$ to u in the *special space* $H^{s(\alpha)}(\overline{\Omega})$, adapted to the fractional Dirichlet problem, for $\alpha > 1/2$ [s-transmission property, Hörmander 1965, Grubb 2015]. One has

$$H^{\alpha}_{\mathrm{comp}}(\Omega) \subset H^{s(\alpha)}(\overline{\Omega}) \subset H^{\alpha}_{\mathrm{loc}}(\Omega)$$

but solutions in $H^{s(\alpha)}(\overline{\Omega})$ may have singularities near $\partial\Omega$.

Summary

1. The *Runge property* for second order PDE allows to approximate solutions in $U \subset \Omega$ using solutions in Ω.
2. Runge approximation is useful in the Calderón problem under monotonicity conditions. For general smooth coefficients, need complex geometrical optics.
3. The fractional operator $(-\Delta)^s$, $0 < s < 1$, is *nonlocal*. The DN map takes exterior Dirichlet values $u|_{\Omega_e}$ to exterior Neumann values $(-\Delta)^s u|_{\Omega_e}$.
4. Fractional equations may have *strong uniqueness and approximation properties*, replacing complex geometrical optics and leading to strong results in inverse problems.