PSEUDODIFFERENTIAL SYMBOLS

JESSE RAILO

ABSTRACT. This note is for the reading group of inverse problems at the University of Jyväskylä in Autumn 2017. The reading group will discuss microlocal analysis and its applications. This note contains 50 minutes talk on pseudodifferential symbols. It is based on *Microlocal analysis and evolution equations* (J. Wunsch) chapter 3, and *An introduction to pseudo-differential operators* (M. Wong) chapter 4. I thank everyone who pointed out some mistakes during my talk, which I have now (hopefully) corrected here.

1. SYMBOLS OF LINEAR PARTIAL DIFFERENTIAL OPERATORS

We call the operator

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha, \quad D_j = \frac{1}{i}\partial_{x^j},$$

a *linear differential operator of order* m and denote $P \in \mathrm{Diff}^m(X)$. Here $X, Y = \mathbb{R}^n$ for some $n \in \mathbb{Z}_+$, α, β are multi-indices, etc. We assume that the coefficients are smooth $a_\alpha(x) \in C^\infty(X)$. In Riemannian manifolds one would need to define similar concepts in local coordinates and check that definitions are coordinate invariant (behave nicely under changes of variables). We do not pursue much to that direction in this note.

We define the *total symbol* of P as

$$\sigma_{\mathrm{tot}}(P)(x, \xi) := p(x, \xi) := \sum_{|\alpha| \leq m} a_\alpha(x)\xi^\alpha.$$

It is a bijective mapping from differential operators to the polynomials of ξ with smooth coefficients. However, since differential operators do not commute while polynomials do, it is not a ring homomorphism. We define the *principal symbol* of P as $\sigma_m(P) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ to be the top-order part of p.

In this short introductory chapter we describe some properties of the principal symbols which make them nice in comparison to the total symbol which is often too technical to manipulate in practice.

One easily notices that $\sigma_m(P)$ is a homogeneous polynomial of degree m in ξ, and that $\sigma_m(P)(x,\lambda\xi) = \lambda^m \sigma_m(P)(x,\xi)$ for any $\lambda \in \mathbb{R}$. Hence it is also sufficient to know $\hat{\sigma}_m(P) := \hat{\sigma}_m(P)|_{|\xi|=1}$. We also point that one can think symbols as functions $\sigma_{\mathrm{tot}}(P), \sigma_m(P) : T^*X \to \mathbb{R}$ and $\hat{\sigma}_m(P) : S^*X \to \mathbb{R}$. In the Euclidean case $T^*X = \mathbb{R}^n \times \mathbb{R}^n$ and $S^*X = \mathbb{R}^n \times S^{n-1}$.

Proposition 1.1. *Let $P \in \mathit{Diff}^N(X), Q \in \mathit{Diff}^M(X)$. Then*

$$\sigma_{M+N}(PQ) = \sigma_N(P)\sigma_M(Q).$$

Proof. Write

$$P = \sum_{|\alpha|\leq N} a_\alpha(x) D^\alpha, \quad Q = \sum_{|\beta|\leq M} b_\beta(x) D^\beta.$$

In calculations we denote by R a differential operator of non-maximal order, and it may vary from line to line. We calculate that

$$\begin{aligned} PQ &= \left(\sum_{|\alpha|=N} a_\alpha(x) D^\alpha\right)\left(\sum_{|\beta|=M} b_\beta(x) D^\beta\right) + R \\ &= \sum_{|\alpha|=N}\sum_{|\beta|=M} a_\alpha(x) D^\alpha(b_\beta(x) D^\beta) + R \\ &= \sum_{|\alpha|=N}\sum_{|\beta|=M} a_\alpha(x)\left(\sum_{\alpha'\leq\alpha}\binom{\alpha}{\alpha'} D^{\alpha'} b_\beta(x) D^{\alpha-\alpha'}\right) D^\beta + R \\ &= \sum_{|\alpha|=N}\sum_{|\beta|=M} a_\alpha(x) b_\beta(x) D^{\alpha+\beta} + R, \end{aligned}$$

where in the second last step we used the generalized Leibniz rule for multi-index derivatives, and in the last step the fact that when ever derivative $D^{\alpha'}$, $\alpha \neq 0$, hits $b_\beta(x)$, then the order of the corresponding term is less than $M+N$. □

We remark that the above property does not hold for the total symbol, it holds for $\hat{\sigma}_m$, and shows also that $[P,Q]$ has the order at most $M+N-1$.

Let $\phi : X \to Y$ be a diffeomorphism (with $X = Y = \mathbb{R}^n$). We define the *pullback* of $f : Y \to \mathbb{R}$ by $\phi^* f(x) = f(\phi(x))$. If P is a differential operator on $C^\infty(X)$, we define its *pushforward* as

$$(\phi_* P)(f)(y) = P(\phi^* f)(\phi^{-1}(y)).$$

Notice that the pushforward $\phi_* P$ is a differential operator on $C^\infty(Y)$.

Proposition 1.2. *Let $P \in \mathit{Diff}^m(X)$ and $\phi : X \to Y$ be a diffeomorphism. Then*

$$\sigma_m(\phi_* P)(y, \eta) = \sigma_m(P)(\phi^{-1}(y), d\phi(\eta))$$

where $d\phi$ denotes the Jacobian of ϕ.

Proof. We need to calculate the principal symbol of $\phi_* P$, and hence we calculate $\phi_* P$. Denote $y = \phi(x)$. We do some preparational calculations. Let $f : Y \to \mathbb{R}$. Now

$$\begin{aligned} \phi_*(D_{x^j})(f)(y) &= D_{x^j}(f(\phi(x)))|_{x=\phi^{-1}(y)} \\ &= \sum_{i=1}^n \frac{d\phi^i(x)}{dx^j}(D_{y^i} f)(\phi(x))|_{x=\phi^{-1}(y)} \\ &= \sum_{i=1}^n \frac{dy^i}{dx^j}(D_{y^i} f)(y). \end{aligned}$$

Hence $\phi_* D_{x^j} = \sum_{i=1}^n \frac{dy^i}{dx^j} D_{y^i}$.

We use the formula above and

$$\phi_*(D_x^\alpha)(f)(y) = D_{x^1}^{\alpha_1} \cdots D_{x^n}^{\alpha_n}(f(\phi(x))|_{x=\phi^{-1}(y)}$$

to find that

$$\phi_*(D_x^\alpha) = D_{x^1}^{\alpha_1} \cdots D_{x^n}^{\alpha_n - 1} \sum_{i=1}^n \frac{dy^i}{dx^n} D_{y^i} f(\phi(x))|_{x=\phi^{-1}(y)}.$$

We are interested only on the highest order. Hence, whenever the derivatives $D_{x^1}^{\alpha_1} \cdots D_{x^n}^{\alpha_n - 1}$ hit to the coefficients coming from the diffeomorphism ϕ the corresponding term is not of the highest order, and when it hits to the term $D_{y^i} f(\phi(x))|_{x=\phi^{-1}(y)}$, then one gets term of the form

$$\sum_{i=1}^n \frac{dy^i}{dx^j} \sum_{k=1}^n \frac{dy^k}{dx^l} D_{y^k} D_{y^i}$$

for some $l = 1, \ldots, n$. We remark that the symbol of such terms can be written simply as a product

$$\sum_{i=1}^n \frac{dy^i}{dx^j} \eta_i \cdot \sum_{k=1}^n \frac{dy^k}{dx^l} \eta_k.$$

We deduce, inductively, that

$$\begin{aligned} \sigma_m(\phi_* P)(y, \eta) &= \sum_{|\alpha|=m} a_\alpha(\phi^{-1}(y)) \prod_{j=1}^n \left(\sum_{i=1}^n \frac{dy^i}{dx^j} \eta_i \right)^{\alpha_j} \\ &= \sigma_m(P)(\phi^{-1}(y), d\phi(\eta)) \end{aligned}$$

as claimed.

□

We remark that the above property does not hold for the total symbol, and it holds for $\hat{\sigma}_m$.

2. Kohn-Nirenberg symbols and classical symbols

The reader should be familiar with the Fourier transform and tempered distributions. One could consult for example the chapters 1–3 of the book by M. Wong, to get a rapid course on the topic. We define two symbol classes: $S^m = S^m_{KN}$ Kohn-Nirenberg symbols and S^m_{cl} classical symbols. We give examples of symbols within those classes, discuss on their relation in the compact case and explain why the pseudodifferential operators (integral operators) of the symbols are well-defined.

Denote $\langle \xi \rangle := (1+|\xi|^2)^{1/2}$ and let $m \in \mathbb{R}$. We say that $a \in C^\infty(T^*X)$ is a *Kohn-Nirenberg symbol of order* m if for all α, β holds

$$\sup_{(x,\xi)\in T^*X} \langle \xi \rangle^{|\beta|-m} \left| \partial_x^\alpha \partial_\xi^\beta a \right| = C_{\alpha,\beta} < \infty.$$

We then denote that $a \in S^m = S^m_{KN}$.

We remark that $a \in S^m$ if and only if $\forall \alpha, \beta \in \mathbb{N}^n : \exists C_{\alpha,\beta} < \infty :$ $\left| \partial_x^\alpha \partial_\xi^\beta a \right| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}$ for all $(x,\xi) \in T^*X$ (this is a trivial observation!). One could also replace $\langle \xi \rangle$ by $1 + |\xi|$ since they behave asymptotically similarly at infinity (and at zero); it however may change the optimal coefficients.

Next we give two examples of symbols satisfying this condition.

Example 2.1. Let $p(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$ where $a_\alpha \in C^\infty(X)$. If the derivatives of all order of a_α are bounded for any $|\alpha| \leq m$, then $p \in S^m$.

Proof. Let γ, δ be multi-indices. Then for any $(x,\xi) \in T^*X$ it holds that

$$\left|(D_x^\gamma D_\xi^\delta p)(x,\xi)\right| \leq \sum_{|\alpha|\leq m} C_{\alpha,\gamma} \left|\partial_\xi^\delta \xi^\alpha\right|$$

where $C_{\alpha,\gamma} = \sup_{x\in\mathbb{R}^n} |(D_x^\gamma a_\alpha)(x)|$. Notice that $C_{\alpha,\gamma}$ is finite by assumption. A direct calculation shows that

$$\partial_\xi^\delta \xi^\alpha = \delta! \binom{\alpha}{\delta} \xi^{\alpha-\delta} \chi_{\delta\leq\alpha}$$

for all $\xi \in \mathbb{R}^n$.

Hence

$$\left|(D_x^\gamma D_\xi^\delta p)(x,\xi)\right| \leq \sum_{|\alpha|\leq m} C_{\alpha,\gamma}\delta!\binom{\alpha}{\delta}|\xi|^{|\alpha|-|\delta|}$$
$$\leq C_{\gamma,\delta}\langle\xi\rangle^{m-|\delta|}$$

where $C_{\gamma,\delta} = \sum_{|\alpha|\leq m} C_{\alpha,\gamma}\delta!\binom{\alpha}{\gamma}$. We also used the fact that $|\xi| \leq (1+|\xi|^2)^{1/2} = \langle\xi\rangle$. □

Example 2.2. Let $\omega(x,\xi) = (1+|\xi|^2)^{m/2}, -\infty < m < \infty$. Then $\omega \in S^m$.

Proof. It is enough to prove that $\left|D^\beta\sigma(x)\right| \leq C_{m,\beta}\langle\xi\rangle^{m-|\beta|}$ for some $C_{m,\beta} < \infty$. This is true if $\beta = 0$ with coefficient 1 (in fact identity holds!). We continue by induction, and assume that for any $m \in (-\infty,\infty)$ and multi-indices β of lengths at most equal to L the claim is true. Let γ be a multi-index of length $L+1$. Then $D^\gamma = D^\beta D_j$ for some $j = 1,\ldots,n$ and some multi-index β of length L.

Let us define a function $\tau(\xi) = m\xi_j(1+|\xi|^2)^{m/2-1}$. Now it holds that

$$|(D^\gamma\omega)(\xi)| = \left|(D^\beta D_j\omega)(\xi)\right| = \left|(D^\beta\tau)(\xi)\right|.$$

By the generalized Leibniz rule

$$(D^\beta\tau)(\xi) = m\sum_{\beta'\leq\beta}\binom{\beta}{\beta'}(D^{\beta'}\xi_j)D^{\beta-\beta'}(1+|\xi|^2)^{m/2-1}.$$

By the induction assumption $(1+|\xi|^2)^{m/2-1}$ is a symbol of order $m-2$, and by the previous example considering symbols of differential operators ξ_j is a symbol of order 1. Hence we have

$$\left|(D^\beta\tau)(\xi)\right| \leq C'_{m,\beta}\sum_{\beta'\leq\beta}\binom{\beta}{\beta'}\langle\xi\rangle^{1-|\beta'|}\langle\xi\rangle^{m-2-|\beta|+|\beta'|}$$

for some finite $C'_{m,\beta}$. Hence

$$\left|(D^\beta\tau)(\xi)\right| \leq C_{m,\beta}\langle\xi\rangle^{m-|\gamma|}$$

where $C_{m,\beta} = C'_{m,\beta}\sum_{\beta'\leq\beta}\binom{\beta}{\beta'}$. We remark that β was defined by γ, and this completes the proof. □

We say that $a \in C^\infty(T^*X)$ is a *classical symbol of order* m if there exists $\tilde{a} \in C^\infty(S^*X\times\mathbb{R}_+)$ such that for all $\xi \in T^*X$ with $|\xi| > 1$ it holds that

$$a(x,\xi) = |\xi|^m\,\tilde{a}(x,\hat{\xi},|\xi|^{-1})$$

where $\hat{\xi} = \xi/|\xi|$. We then denote that $a \in S^m_{cl}(T^*X)$.

Proposition 2.3. *Let $S^m_{cl,c}$ denote the symbols of that have compact support in the base variable x (uniformly with respect to ξ). Then $S^m_{cl,c} \subset S^m$.*

Proof. Exercise 3.4. in the notes of Wunsch. We leave it to the reader. ◻

Next we define the pseudodifferential operator for a given symbol. Let $a \in S^m$ for some $m \in \mathbb{R}$. Then the *pseudodifferential operator associated to a* is defined by

$$(\mathrm{Op}(a)f)(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} a(x,\xi)\hat{f}(\xi)d\xi$$

for any $f \in S(\mathbb{R}^n)$. Here $S(\mathbb{R}^n)$ denote the Schwartz space (test functions) and $\hat{f}$ is the Fourier transform of f defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-i\langle x,\xi\rangle}dx.$$

The Fourier inversion formula for test functions shows that the pseudodifferential operator associated to the symbol of a linear partial differential operator is the linear partial differential operator itself. One can also show that the operator associated to $(1+|\xi|^2)^{m/2}$ is in fact $(I-\Delta)^{m/2}$ where $\Delta = \sum_{j=1}^n \partial^2_{x^j}$ is the Laplacian. These are based on properties of the Fourier transform (see also the last proposition of this note).

The following two propositions justify the definition of the symbol class S^m and pseudodifferential operators based on it.

Proposition 2.4. *If $a, b \in S^m$ and $Op(a) = Op(b)$, then $a = b$.*

Proof. See e.g. Proposition 4.5. and its proof in the book of Wong. This is based on the Fourier inversion formula and the property that if f is a continuous tempered function such that $\langle f, \phi\rangle = 0$ for every test function, i.e. the distribution $D(f) \equiv 0$, then $f \equiv 0$. (A measurable function f is said to be *tempered* if $\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^N}dx < \infty$ for some $N \in \mathbb{Z}_+$.) ◻

Proposition 2.5. *If $a \in S^m$, then $Op(a)$ is a mapping from $S(\mathbb{R}^n)$ to itself.*

Proof. See e.g. Proposition 4.7. and its proof in the book of Wong. ◻

Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD) FI-40014 University of Jyväskylä, Finland
E-mail address: jesse.t.railo@jyu.fi