Analysis on manifolds

Exercises, 23.10.2014

(return by Thu 27.11.2014)

1. Prove Theorem 2.2 in the lecture notes.
2. Prove Lemma 2.6 in the lecture notes.
3. Let $U \subset \mathbb{R}^n$ be star-shaped and let $\alpha \in \Omega^1(U)$ satisfy $d\alpha = 0$. Show directly that $\alpha = df$ for some $f \in \Omega^0(U)$.
4. Show that any inner product $\langle \,\cdot\, , \,\cdot\, \rangle$ on $\mathbb{R}^n$ is given by
$$\langle v, w\rangle = Av \cdot w, \qquad v, w \in \mathbb{R}^n$$
for some positive definite symmetric matrix $A = (a_{jk})_{j,k=1}^n$, and conversely any such A determines an inner product by the above formula.
5. (Exercise 2.1) Show that $L_g(\gamma)$ is independent of the way the curve γ is parametrized, and that we may always parametrize γ by *arc length* so that $|\dot\gamma(t)|_g = 1$ for all t.
6. (Exercise 2.2) Show that d_g is a metric distance function on U, and that (U, d_g) is a metric space whose topology is the same as the Euclidean topology on U.
7. Verify that $\Gamma_{jk}^l = \Gamma_{kj}^l$ and $\partial_k g_{ij} = \Gamma_{ki}^l g_{lj} + \Gamma_{kj}^l g_{il}$.
8. Generalize the first variation formula (Lemma 2.12) to the case of a variation that does not fix the endpoints of the curve.
9. Show that the L^2 inner product on m-tensor fields in (U, g) is indeed an inner product.
10. If $\beta \in \Omega^k(U)$, show that $\delta\beta = \gamma_I \, dx^I$ where
$$\tilde\gamma_{i_1 \ldots i_{k-1}}(x) = -g^{lr}(\partial_r \tilde\beta_{l i_1 \ldots i_{k-1}} - \Gamma_{lr}^j \tilde\beta_{j i_1 \ldots i_{k-1}}).$$
11. Show that $\Delta_g(uv) = (\Delta_g u)v + 2\langle du, dv\rangle_g + u(\Delta_g v)$ for $u, v \in C^2(U)$.
12. If $U \subset \mathbb{R}^n$ is a bounded open set with C^1 boundary, and if $u \in C^2(\overline{U})$ satisfies $\Delta_g u = 0$ in U and $u|_{\partial U} = 0$, show that $u = 0$.