

Unique continuation for elliptic equations

Notes, Fall 2014

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Preface

The inverse problems reading group this fall will begin by studying the unique continuation principle for elliptic partial differential equations. This principle, which states that any solution of an elliptic equation that vanishes in a small ball must be identically zero, is a fundamental property that has various applications e.g. in solvability questions, inverse problems, and control theory.

Possible topics to be covered:

1. Overview
2. Real-analytic coefficients (Holmgren's theorem)
3. L^2 Carleman inequalities
4. Doubling/three spheres inequalities
5. Frequency function method
6. L^p Carleman inequalities
7. The 2D case
8. Counterexamples to unique continuation
9. Pseudoconvexity for general operators
10. Nonlinear equations

CHAPTER 1

Introduction

The purpose of these notes is to discuss the unique continuation principle (UCP) for elliptic second order partial differential equations. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set. For the most part we will consider linear operators

$$Pu = a^{jk} \partial_{jk} u + b^j \partial_j u + cu,$$

where the coefficients satisfy

$$a^{jk} \in W^{1,\infty}(\Omega), \quad b^j \in L^\infty(\Omega), \quad c \in L^\infty(\Omega),$$

and (a^{jk}) is a symmetric matrix satisfying the uniform ellipticity condition for some constant $\lambda > 0$,

$$a^{jk}(x) \xi_j \xi_k \geq \lambda |\xi|^2 \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$

A simple example to keep in mind is the elliptic Schrödinger operator $P = -\Delta + q$ where $q \in L^\infty(\Omega)$.

The UCP comes in several different forms:

THEOREM 1.1. (*Weak UCP*) If $u \in H^2(\Omega)$ satisfies

$$Pu = 0 \quad \text{in } \Omega$$

and

$$u = 0 \text{ in some ball } B \text{ contained in } \Omega,$$

then $u = 0$ in Ω .

THEOREM 1.2. (*Strong UCP*) If $u \in H^2(\Omega)$ satisfies

$$Pu = 0 \quad \text{in } \Omega$$

and if u vanishes to infinite order at $x_0 \in \Omega$ in the sense that

$$\lim_{r \rightarrow 0} \frac{1}{r^N} \int_{B(x_0, r)} |u|^2 dx = 0 \quad \text{for all } N \geq 0,$$

then $u = 0$ in Ω .

THEOREM 1.3. (*UCP across a hypersurface*) Let S be a C^∞ hypersurface such that $\Omega = S_+ \cup S \cup S_-$ where S_+ and S_- denote the two sides of S . If $x_0 \in S$ and if V is an open neighborhood of x_0 in Ω , and if $u \in H^2(V)$ satisfies

$$Pu = 0 \quad \text{in } V$$

and

$$u = 0 \quad \text{in } V \cap S_-,$$

then $u = 0$ in some neighborhood of x_0 .

THEOREM 1.4. (*UCP for local Cauchy data*) Let $\Omega \subset \mathbb{R}^n$ have smooth boundary, and let Γ be a nonempty open subset of $\partial\Omega$. If $u \in H^2(\Omega)$ satisfies

$$Pu = 0 \quad \text{in } \Omega$$

and

$$u|_\Gamma = \partial_\nu u|_\Gamma = 0,$$

then $u = 0$ in Ω .

REMARKS.

1. Note that since P is linear, weak UCP implies that any two solutions u_1 and u_2 that coincide in a small ball must be equal in the whole domain. This explains the name "unique continuation principle".
2. Once clearly has

$$\text{strong UCP} \implies \text{weak UCP}.$$

It is also not hard to see that

$$\text{UCP across a hypersurface} \implies \text{weak UCP} \implies \text{UCP for Cauchy data}.$$

3. The above theorems remain valid if $u \in H^1(\Omega)$.
4. In the above theorems, the condition $Pu = 0$ in Ω can be replaced by the differential inequality

$$|a^{jk}\partial_{jk}u| \leq C(|\nabla u| + |u|) \quad \text{a.e. in } \Omega.$$

5. The assumption that the coefficients (a^{jk}) are Lipschitz continuous is optimal for $n \geq 3$, in the sense that for any $\alpha < 1$ there exist C^α coefficients (a^{jk}) such that UCP does not hold for the corresponding operator. (If $n = 2$ the UCP holds for $a^{jk} \in L^\infty$.) However, the assumptions for the first and zeroth order terms can be improved, and in fact UCP holds if $b^j \in L^n(\Omega)$ and $c \in L^{n/2}(\Omega)$ (at least when $n \geq 3$).

6. The UCP for uniformly elliptic nonlinear equations often reduces to the linear case.

The unique continuation principle is a fundamental property of elliptic equations. The UCP and the methods used for studying it, in particular Carleman inequalities, have various applications including the following:

- Solvability results for a linear PDE $Au = f$ can often be obtained by duality from uniqueness results for the adjoint equation $A^*u = 0$.
- Similarly, controllability results for a linear PDE $Au = 0$ are often equivalent with certain uniqueness results for the adjoint equation.
- Optimal stability results for the Cauchy problem for elliptic equations are closely related to the UCP.
- The UCP and Carleman inequalities play an important role in various inverse boundary value problems for elliptic and evolution equations.

There are various approaches to obtaining unique continuation results for elliptic equations. The earliest such results were valid for real-analytic coefficients (*Holmgren's uniqueness theorem*). In the general case the UCP can be established via certain inequalities, such as:

1. *Doubling inequalities*, stating that

$$\int_{B_{2r}} u^2 dx \leq C \int_{B_r} u^2 dx \quad \text{for all } u \text{ with } Pu = 0.$$

2. *Three spheres inequalities*, stating that

$$\|u\|_{L^2(B_{2r})} \leq C \|u\|_{L^2(B_r)}^{1-\theta} \|u\|_{L^2(B_{3r})}^{\theta} \quad \text{for all } u \text{ with } Pu = 0,$$

for some θ with $0 < \theta < 1$.

3. *The frequency function method*, which in the case $P = -\Delta$ states that the frequency function

$$F(r) = \frac{r \int_{B_r} |\nabla u|^2 dx}{\int_{\partial B_r} u^2 dS}$$

is increasing with respect to r if u is a harmonic function.

4. *Carleman inequalities*, which state that for any $\tau > 0$ sufficiently large one has

$$\|e^{\tau\varphi}w\|_{L^2(\Omega)} \leq \frac{C}{\tau^\alpha} \|e^{\tau\varphi}Pw\|_{L^2(\Omega)} \quad \text{for all } w \in C_c^\infty(\Omega),$$

where $C > 0$ and $\alpha > 0$ are independent of τ , and φ is a suitable real valued weight function (for instance $\varphi(x) = x_n$).

It is clear that the doubling and three spheres inequalities immediately yield a form of unique continuation, since they imply that any solution that vanishes on B_r must also vanish on B_{2r} . Also the monotonicity of the frequency function related to the Laplace equation (or the corresponding property for general P) imply the UCP. It is not immediately obvious how Carleman inequalities would lead to the UCP, but we will see that they do and in fact they seem to be the most powerful general method for establishing unique continuation properties. In particular, the other inequalities above may be derived from Carleman inequalities, and Carleman inequalities also lead to versions of the UCP for many non-elliptic equations.

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CHAPTER 2

Equations with real-analytic coefficients

CHAPTER 3

Carleman inequalities

3.1. UCP across a hyperplane

We will begin by considering a very simple case, which illustrates the main ideas with minimal technicalities. This case is related to the UCP across a hyperplane for solutions of an elliptic equation in an infinite strip.

THEOREM 3.1. *Let $\Omega = \{x \in \mathbb{R}^n ; a < x_n < b\}$, let $q \in L^\infty(\Omega)$, and assume that $u \in H^2(\Omega)$ solves*

$$(-\Delta + q)u = 0 \text{ in } \Omega.$$

If $u|_{\{b-\varepsilon < x_n < b\}} = 0$ for some $\varepsilon > 0$, then $u \equiv 0$ in Ω .

Since the domain Ω is unbounded, a few remarks are in order.

- (1) The Sobolev spaces on Ω are defined for $k \geq 0$ by

$$H^k(\Omega) := \{u \in L^2(\Omega) ; \partial^\alpha u \in L^2(\Omega) \text{ for } |\alpha| \leq k\}.$$

We define $H_0^k(\Omega)$ to be the closure of $C_c^\infty(\Omega)$ in $H^k(\Omega)$.

- (2) If $q \in L^\infty(\Omega)$, we say that $u \in H^1(\Omega)$ is a weak solution of $(-\Delta + q)u = 0$ in Ω if

$$\int_{\Omega} (\nabla u \cdot \nabla \bar{v} + qu\bar{v}) dx = 0 \quad \text{for all } v \in H_0^1(\Omega).$$

- (3) Now if u is a weak solution of $(-\Delta + q)u = 0$ such that $u|_{\{x_n=a\}} = u|_{\{x_n=b\}} = 0$, so $u \in H_0^1(\Omega)$, then using u as a test function implies

$$\int_{\Omega} (|\nabla u|^2 + q|u|^2) dx = 0.$$

Thus, at least if $q \geq 0$, we obtain $\int_{\Omega} |\nabla u|^2 dx = 0$ which implies $u \equiv 0$ by the boundary condition.

The previous remarks show that any solution vanishing on the whole boundary $\partial\Omega = \{x_n = a, b\}$ is identically zero at least if $q \geq 0$, which follows from uniqueness in the Dirichlet problem. On the other hand,

Theorem 3.1 states that any solution vanishing near one part of the boundary ($\{x_n = b\}$) is identically zero, with no a priori conditions on the other part of the boundary ($\{x_n = a\}$).

Theorem 3.1 will follow from the following a priori L^2 estimate which involves exponential weights.

THEOREM 3.2. (*Carleman inequality for $-\Delta$*) Assume that $\Omega = \{x \in \mathbb{R}^n; a < x_n < b\}$. Then for any $\tau \in \mathbb{R} \setminus \{0\}$ one has

$$\|w\|_{L^2(\Omega)} \leq \frac{C}{|\tau|} \|e^{\tau x_n}(-\Delta)e^{-\tau x_n}w\|_{L^2(\Omega)}, \quad w \in H_0^2(\Omega),$$

where $C = \frac{b-a}{2\pi}$.

A major point in the Carleman inequality is that the constant on the right is $\frac{C}{|\tau|}$ where $C = \frac{b-a}{2\pi}$ is independent of τ . By taking τ very large we can make the constant $\frac{C}{|\tau|}$ small, which makes it possible to absorb various error terms. (We remark that by using a stronger weight, the constant improves to $\frac{C}{|\tau|^{3/2}}$ as we will see later, but then the sign of τ will be important.)

As an example of absorbing errors by taking τ large, we can easily include an L^∞ potential in the Carleman inequality.

THEOREM 3.3. (*Carleman inequality for $-\Delta + q$*) Assume that $\Omega = \{x \in \mathbb{R}^n; a < x_n < b\}$, and let $q \in L^\infty(\Omega)$. If $|\tau| > \frac{b-a}{\pi} \|q\|_{L^\infty(\Omega)}$, one has

$$\|w\|_{L^2(\Omega)} \leq \frac{2C}{|\tau|} \|e^{\tau x_n}(-\Delta + q)e^{-\tau x_n}w\|_{L^2(\Omega)}, \quad w \in H_0^2(\Omega),$$

where $C = \frac{b-a}{2\pi}$.

PROOF. Theorem 3.2 gives

$$\begin{aligned} \|w\|_{L^2(\Omega)} &\leq \frac{C}{|\tau|} \|e^{\tau x_n}(-\Delta)e^{-\tau x_n}w\|_{L^2(\Omega)} \\ &\leq \frac{C}{|\tau|} \|e^{\tau x_n}(-\Delta + q)e^{-\tau x_n}w\|_{L^2(\Omega)} + \frac{C\|q\|_{L^\infty(\Omega)}}{|\tau|} \|w\|_{L^2(\Omega)} \\ &\leq \frac{C}{|\tau|} \|e^{\tau x_n}(-\Delta + q)e^{-\tau x_n}w\|_{L^2(\Omega)} + \frac{1}{2} \|w\|_{L^2(\Omega)} \end{aligned}$$

by using the condition for τ . □

Before proving Theorem 3.2, let us give the main argument that allows to obtain the unique continuation result (Theorem 3.1) from the Carleman inequality.

PROOF OF THEOREM 3.1 GIVEN THEOREM 3.2. We assume that $\Omega = \{x \in \mathbb{R}^n; a < x_n < b\}$ and $u \in H^2(\Omega)$ satisfies

$$\begin{aligned} (-\Delta + q)u &= 0 \text{ in } \Omega, \\ u &= 0 \text{ in } \{b - \varepsilon < x_n < b\}. \end{aligned}$$

Let c_0 be any number satisfying $a < c_0 < b$. We want to show that $u = 0$ in $\{c_0 < x_n < b\}$.

We rewrite the estimate from Theorem 3.3 as

$$\|e^{\tau x_n} w\|_{L^2(\Omega)} \leq \frac{C}{\tau} \|e^{\tau x_n} (-\Delta + q)w\|_{L^2(\Omega)}$$

which holds for all $w \in H_0^2(\Omega)$ if $\tau > 0$ is sufficiently large. Now choose

$$w = \chi u$$

where $\chi(x', x_n) = \zeta(x_n)$ where $\zeta \in C^\infty(\mathbb{R})$ satisfies $\zeta = 1$ for $t \geq c_0$ and $\zeta = 0$ near $t \leq a$. Using that $\chi = 0$ near $\{x_n = a\}$ and $u = 0$ near $\{x_n = b\}$, we have $w \in H_0^2(\Omega)^1$ and therefore

$$\begin{aligned} \|e^{\tau x_n} u\|_{L^2(\{c_0 < x_n < b\})} &\leq \|e^{\tau x_n} \chi u\|_{L^2(\Omega)} \\ &\leq \frac{C}{\tau} \|e^{\tau x_n} (-\Delta + q)(\chi u)\|_{L^2(\Omega)} \\ &\leq \frac{C}{\tau} (\|e^{\tau x_n} \chi (-\Delta + q)u\|_{L^2(\Omega)} + \|e^{\tau x_n} [\Delta, \chi]u\|_{L^2(\Omega)}) \end{aligned}$$

where $[\Delta, \chi]v = 2\nabla\chi \cdot \nabla v + (\Delta\chi)v$. In particular $[\Delta, \chi]u$ is supported in $\text{supp}(\nabla\chi) \subset \{a \leq x_n \leq c_0\}$. Using that $(-\Delta + q)u = 0$, the previous inequality implies

$$\|e^{\tau x_n} u\|_{L^2(\{c_0 < x_n < b\})} \leq \frac{C}{\tau} \|e^{\tau x_n} [\Delta, \chi]u\|_{L^2(\{a < x_n < c_0\})}.$$

But $e^{\tau x_n}$ is $\geq e^{\tau c_0}$ when $x_n \geq c_0$ and $\leq e^{\tau c_0}$ when $x_n \leq c_0$. This implies that

$$\begin{aligned} e^{\tau c_0} \|u\|_{L^2(\{c_0 < x_n < b\})} &\leq \|e^{\tau x_n} u\|_{L^2(\{c_0 < x_n < b\})} \\ &\leq \frac{C}{\tau} \|e^{\tau x_n} [\Delta, \chi]u\|_{L^2(\{a < x_n < c_0\})} \\ &\leq \frac{C}{\tau} e^{\tau c_0} \|[\Delta, \chi]u\|_{L^2(\{a < x_n < c_0\})}. \end{aligned}$$

But $[\Delta, \chi]u$ is a fixed function, so dividing by $e^{\tau c_0}$ and taking $\tau \rightarrow \infty$ shows that $\|u\|_{L^2(\{c_0 < x_n < b\})} = 0$. Thus $u|_{\{c_0 < x_n < b\}}$ as required. \square

¹Justify this.

REMARK. Draw a picture explaining the regions where x_n has the right sign.

We will now prove Theorem 3.2. The main point in the proof is a positive commutator argument (based on integration by parts) that gives a lower bound for the conjugated Laplacian $e^{\tau x_n}(-\Delta)e^{-\tau x_n}$ using its decomposition to self-adjoint and skew-adjoint parts. The proof also uses the Poincaré inequality in a strip (Theorem 3.4 below).

PROOF OF THEOREM 3.2. Write $P := e^{\tau x_n}(-\Delta)e^{-\tau x_n}$. By density, it is enough to prove the estimate

$$\|w\| \leq \frac{C}{\tau} \|Pw\|, \quad w \in C_c^\infty(\Omega),$$

where $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$. Writing $D = -i\nabla$, note that

$$e^{\tau\varphi}D(e^{-\tau\varphi}w) = (D + i\tau\nabla\varphi)w.$$

It follows that

$$P = e^{\tau x_n}De^{-\tau x_n} \circ e^{\tau x_n}De^{-\tau x_n} = (D + i\tau e_n)^2 = D^2 + 2i\tau D_n - \tau^2.$$

We want a lower bound for $\|Pw\|$, and to achieve this we write P in terms of its self-adjoint and skew-adjoint parts, i.e. $P = A + iB$ where A and B are the formally self-adjoint operators

$$A = \frac{P + P^*}{2}, \quad B = \frac{P - P^*}{2i}.$$

Here P^* is the formal adjoint of P given by

$$P^* = D^2 - 2i\tau D_n - \tau^2.$$

It follows that

$$A = D^2 - \tau^2, \quad B = 2\tau D_n.$$

Now if $w \in C_c^\infty(\Omega)$, we write $(\cdot, \cdot) = (\cdot, \cdot)_{L^2(\Omega)}$ and compute

$$\begin{aligned} \|Pw\|^2 &= ((A + iB)w, (A + iB)w) \\ &= \|Aw\|^2 + \|Bw\|^2 + i(Bw, Aw) - i(Aw, Bw) \\ (3.1) \quad &= \|Aw\|^2 + \|Bw\|^2 + i([A, B]w, w) \end{aligned}$$

where $[A, B] := AB - BA$ is the commutator of A and B . In the last line we integrated by parts, using that $w \in C_c^\infty(\Omega)$.

The computation (3.1) shows that we may expect a lower bound for $\|Pw\|^2$ provided that the commutator $i[A, B]$ is positive, in the

sense that $i([A, B]w, w) \geq 0$. In our case both A and B are constant coefficient operators, and therefore $[A, B] \equiv 0$. Thus

$$\|Pw\|^2 = \|Aw\|^2 + \|Bw\|^2.$$

Here $B = 2\tau\partial_n$ so it is enough to have a lower bound for $\|\partial_n w\|^2$. This follows from the Poincaré inequality in a strip, which is Theorem 3.4 below. Forgetting the $\|Aw\|^2$ term, we have

$$\|Pw\|^2 \geq 4\tau^2 \|\partial_n w\|^2 \geq \frac{4\tau^2 \pi^2}{(b-a)^2} \|w\|^2$$

which is the required result. \square

THEOREM 3.4. (*Poincaré inequality in a strip*) Assume that $\Omega = \{x \in \mathbb{R}^n; a < x_n < b\}$. Then

$$\int_{\Omega} |u|^2 dx \leq \frac{(b-a)^2}{\pi^2} \int_{\Omega} |\partial_n u|^2 dx, \quad u \in H_0^1(\Omega).$$

The constant is optimal.

PROOF. We start by proving that

$$(3.2) \quad \int_a^b |f(t)|^2 dt \leq \frac{(b-a)^2}{\pi^2} \int_a^b |f'(t)|^2 dt, \quad f \in H_0^1((a, b)).$$

By scaling we may assume that $a = 0$ and $b = \pi$. Let $f \in H_0^1((0, \pi))$ and define

$$h(t) := \begin{cases} f(t), & 0 < t < \pi, \\ -f(-t), & -\pi < t < 0. \end{cases}$$

Then h is an odd function in $H_0^1((-\pi, \pi))$ and thus has a Fourier series

$$h(t) = \sum_{k=-\infty}^{\infty} \hat{h}(k) e^{ikt} = \sum_{k=1}^{\infty} \hat{h}(k) (e^{ikt} - e^{-ikt}).$$

The Parseval identity and the fact that $(h')^{\wedge}(k) = ik\hat{h}(k)$ imply that

$$\begin{aligned} \int_0^{\pi} |f|^2 &= \frac{1}{2} \int_{-\pi}^{\pi} |h|^2 = \pi \sum_{k \neq 0} |\hat{h}(k)|^2 \\ &\leq \pi \sum_{k \neq 0} |ik\hat{h}(k)|^2 = \frac{1}{2} \int_{-\pi}^{\pi} |h'|^2 = \int_0^{\pi} |f'|^2. \end{aligned}$$

This implies (3.2), and it also follows that equality holds in (3.2) iff $f(t) = c \sin(\pi \frac{t-a}{b-a})$ for some constant c .

Now if $u \in C_c^\infty(\Omega)$, the inequality (3.2) gives

$$\int_{\Omega} |u|^2 = \int_{\mathbb{R}^{n-1}} \int_a^b |u|^2 \leq \frac{(b-a)^2}{\pi^2} \int_{\mathbb{R}^{n-1}} \int_a^b |\partial_n u|^2$$

and the result follows by density. The optimality of the constant follows by taking $u(x', x_n) = \varphi(x') \sin(\pi \frac{t-a}{b-a})$ for some $\varphi \in C_c^\infty(\mathbb{R}^{n-1})$. \square

3.2. UCP across a hypersurface

In the previous section, we discussed a simple unique continuation result across a hyperplane $\{x_n = c\}$ in the case where the solution u vanishes in some infinite strip on one side of $\{x_n = c\}$. In this section we will prove a local result, stating that if a solution vanishes on one side of a (not necessarily flat) hypersurface near some point x_0 , then the solution vanishes near x_0 . For simplicity, we restrict our attention to elliptic operators of the form $-\Delta + q$.

THEOREM 3.5. (*UCP across a hypersurface*) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $q \in L^\infty(\Omega)$. Assume that $x_0 \in \Omega$, let V be a neighborhood of x_0 , and let S be a C^∞ hypersurface through x_0 such that $V = V_+ \cup S \cup V_-$ where V_+ and V_- denote the two sides of S . If $u \in H^2(V)$ satisfies*

$$\begin{aligned} (-\Delta + q)u &= 0 & \text{in } V, \\ u &= 0 & \text{in } V_+, \end{aligned}$$

then $u = 0$ in some neighborhood of x_0 .

In Theorem 3.1 we proved the UCP across $\{x_n = c\}$ via Carleman inequalities with weight $\varphi(x) = x_n$. Note that $\{x_n = c\} = \varphi^{-1}(c)$ is a level set of the weight φ . Now the level sets of any hypersurface S are of the form $\varphi^{-1}(c)$ for a suitable φ . Thus it is natural to study more general (and "stronger") weights φ than the linear one. This will also be very useful for localizing the estimates and for considering more general operators.

Given a hypersurface $S = \varphi^{-1}(c)$, there are many functions having S as a level set (any function of the form $f(\varphi)$ has this property). The next result shows that if one starts with any function φ with nonvanishing gradient, the "convexified" weight function $\psi = e^{\lambda\varphi}$ for λ sufficiently large will enjoy a good Carleman inequality.

THEOREM 3.6. (*Carleman inequality with weight $e^{\lambda\varphi}$*) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and assume that $\varphi \in C^4(\overline{\Omega})$ and $q \in L^\infty(\Omega)$ satisfy $\varphi \geq 0$ in $\overline{\Omega}$, $\nabla\varphi \neq 0$ in $\overline{\Omega}$, and

$$\| |\nabla\varphi|^{-1} \|_{L^\infty(\Omega)} + \sup_{1 \leq j \leq 4} \|\nabla^j \varphi\|_{L^\infty(\Omega)} + \|q\|_{L^\infty(\Omega)} \leq M$$

for some constant $M \geq 1$. Let

$$\psi = e^{\lambda\varphi}.$$

There exist $C_0, \lambda_0, \tau_0 > 0$ only depending on M such that whenever $\lambda > \lambda_0$ and $\tau > \tau_0$, one has

$$\lambda^2 \|w\| + \lambda \tau^{-1} \|\nabla w\| \leq C_0 \tau^{-3/2} \|e^{\tau\psi}(-\Delta + q)e^{-\tau\psi}w\|$$

for any $w \in H_0^2(\Omega)$. (The norms are $L^2(\Omega)$ norms.)

REMARK. The condition $\varphi \geq 0$ is not a real restriction. If φ_0 is any function in $C^4(\overline{\Omega})$ with $\nabla\varphi_0 \neq 0$ in $\overline{\Omega}$, then the function $\varphi = \varphi_0 + C$ for some sufficiently large constant C can be used as a weight function. In most cases φ serves the same purpose as φ_0 .

We will now show how the UCP across a hypersurface follows from the Carleman inequality in Theorem 3.6.

PROOF OF THEOREM 3.5 GIVEN THEOREM 3.6. We first consider the special case where $x_0 = 0$ and $S = \{x_n = 0\}$. Assume that $V = B_{4\delta}$ for some small $\delta > 0$, and suppose that $u \in H^2(V)$ solves

$$\begin{aligned} (-\Delta + q)u &= 0 & \text{in } V, \\ u &= 0 & \text{in } V \cap \{x_n > 0\}. \end{aligned}$$

We want to show that u vanishes in $B_\varepsilon \cap \{x_n < 0\}$ for some $\varepsilon > 0$.

Since the problem is local, it is not sufficient to use a Carleman inequality with weight $\varphi_0(x) = x_n$ as in Section 3.1. Rather, we will consider the slightly bent weight

$$\tilde{\varphi}_0(x', x_n) := x_n - |x'|^2 + \delta^2.$$

The level set $\tilde{\varphi}_0^{-1}(0)$ is the parabola $\{x_n = |x'|^2 - \delta^2\}$. Define the sets

$$\begin{aligned} W_+ &:= \{\tilde{\varphi}_0(x) > 0\} \cap \{x_n < 0\}, \\ W_- &:= \{-\delta^2 < \tilde{\varphi}_0(x) < 0\} \cap \{x_n < 0\}. \end{aligned}$$

If $\delta < 1/\sqrt{2}$, both sets are contained in $B_{2\delta}$ and W_+ contains the set $B_\varepsilon \cap \{x_n < 0\}$ if $\varepsilon = \delta^2$.² Our purpose is to prove the estimate

$$(3.3) \quad \|u\|_{L^2(W_+)} \leq C\tau^{-3/2} \|[\Delta, \chi]u\|_{L^2(W_-)}$$

for a suitable constant C and function χ that are independent of $\tau > \tau_0$. Letting $\tau \rightarrow \infty$ shows that $u|_{W_+} = 0$ and therefore $u|_{B_\varepsilon \cap \{x_n < 0\}} = 0$ as required.

With $\tilde{\varphi}_0$ as above, we define $\tilde{\varphi} := \varphi_0 + C$ for some C such that $\tilde{\varphi} \geq 0$, and define $\tilde{\psi} := e^{\lambda\tilde{\varphi}}$. Note that Theorem 3.6 implies

$$(3.4) \quad \lambda^2 \|e^{\tau\tilde{\psi}}w\|_{L^2(\Omega)} \leq C_0\tau^{-3/2} \|e^{\tau\tilde{\psi}}(-\Delta + q)w\|_{L^2(\Omega)}$$

when $\lambda > \lambda_0$, $\tau > \tau_0$, and $w \in H_0^2(\Omega)$. We will choose

$$w = \chi u$$

where $\chi(x) := \zeta(\frac{\tilde{\varphi}_0(x)}{\delta^2})\eta(\frac{|x|}{4\delta})$, and $\zeta \in C^\infty(\mathbb{R})$, $\eta \in C_c^\infty(\mathbb{R})$ satisfy

$$\begin{aligned} \zeta(t) &= 0 \text{ for } t \leq -1 \text{ and } \zeta(t) = 1 \text{ for } t \geq 0, \\ \eta(t) &= 1 \text{ for } |t| \leq 1/2 \text{ and } \eta(t) = 0 \text{ for } |t| \geq 1. \end{aligned}$$

Since $u = 0$ for $x_n > 0$, it follows that $\text{supp}(w) \subset \overline{W_+ \cup W_-}$ and also $\text{supp}([\Delta, \chi]u) \subset \overline{W_-}$. We also note that $e^{\tau\tilde{\psi}}|_{W_-} \leq e^{\tau c_0} \leq e^{\tau\tilde{\psi}}|_{W_+}$ if $c_0 := \tilde{\psi}(0, \dots, 0, -\delta^2)$. Now applying (3.4) with this w implies

$$\begin{aligned} e^{\tau c_0} \|u\|_{L^2(W_+)} &\leq \|e^{\tau\tilde{\psi}}u\|_{L^2(W_+)} \\ &\leq \|e^{\tau\tilde{\psi}}\chi u\|_{L^2(\Omega)} \\ &\leq C_0\lambda^{-2}\tau^{-3/2} \|e^{\tau\tilde{\psi}}(-\Delta + q)(\chi u)\|_{L^2(\Omega)} \\ &\leq C_0\lambda^{-2}\tau^{-3/2} (\|e^{\tau\tilde{\psi}}\chi(-\Delta + q)u\|_{L^2(\Omega)} + \|e^{\tau\tilde{\psi}}[\Delta, \chi]u\|_{L^2(\Omega)}) \\ &\leq C_0\lambda^{-2}\tau^{-3/2} \|e^{\tau\tilde{\psi}}[\Delta, \chi]u\|_{L^2(W_-)} \\ &\leq C_0\lambda^{-2}\tau^{-3/2} e^{\tau c_0} \|[\Delta, \chi]u\|_{L^2(W_-)}. \end{aligned}$$

Here we used the fact that u is a solution and the support conditions. This proves (3.3), and the theorem follows in the special case where $S = \{x_n = 0\}$.

Finally we consider the case where S is a general C^∞ hypersurface. We may normalize matters so that $x_0 = 0$ and $S \cap V = \varphi_0^{-1}(0) \cap V$ where $\varphi_0 \in C^\infty(\mathbb{R}^n)$ satisfies $\nabla\varphi_0 \neq 0$ on $S \cap \overline{V}$. After a rotation and scaling we may assume that $\nabla\varphi_0(0) = e_n$ (so $T_0S = \{x_n = 0\}$). We

²Draw a picture!

may further assume (after shrinking V) that $V = B_{4\delta}$ for some $\delta \leq \delta_0$ which can be taken very small but fixed. Taylor approximation gives that

$$\varphi_0(x', x_n) = x_n + b(x)|x|^2$$

where $|b(x)| \leq C_0$ in $B_{4\delta_0}$. Thus S looks approximately like $\{x_n = 0\}$ in V if δ is chosen small enough, and the two sides of S are given by

$$V_{\pm} = V \cap \{\pm\varphi_0 > 0\}.$$

After these normalizations, we set

$$\tilde{\varphi}_0(x', x_n) := \varphi_0(x) - |x'|^2 + \delta^2.$$

With this choice of $\tilde{\varphi}_0$, we may repeat the argument given above for the case $S = \{x_n = 0\}$ (replacing the sets $\{\pm x_n > 0\}$ by $\{\pm\varphi_0(x) > 0\}$). Since the geometric picture is close to the case where $x_n = 0$, the same argument will show³ that $u|_{B_\varepsilon \cap \{\varphi_0 < 0\}} = 0$ for some $\varepsilon > 0$. This proves the theorem. \square

It remains to prove Theorem 3.6. We begin by introducing some notation. Let (\cdot, \cdot) be the inner product in $L^2(\Omega)$ and $\|\cdot\|$ the corresponding norm, and let $P_0 = D^2$ be the Laplacian where $D = -i\nabla$. If $\psi \in C^4(\overline{\Omega})$ is a real valued function and if $\tau > 0$, we define the conjugated Laplacian

$$P_{0,\psi} = e^{\tau\psi} P_0 e^{-\tau\psi}.$$

We also write ψ'' for the Hessian matrix

$$\psi''(x) = [\partial_{x_j x_k} \psi(x)]_{j,k=1}^n.$$

The next result is analogous to the computation in Theorem 3.2 but involves a more general weight function.

THEOREM 3.7. (Commutator) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $\psi \in C^4(\overline{\Omega})$. Then*

$$P_{0,\psi} = A + iB$$

where A and B are the formally self-adjoint operators

$$\begin{aligned} A &= D^2 - \tau^2 |\nabla \psi|^2, \\ B &= \tau [\nabla \psi \circ D + D \circ \nabla \psi]. \end{aligned}$$

³Check this!

If $w \in H_0^2(\Omega)$ one has

$$\|P_{0,\psi}w\|^2 = \|Aw\|^2 + \|Bw\|^2 + (i[A, B]w, w)$$

where the commutator $i[A, B]$ satisfies

$$(i[A, B]w, w) = 4\tau(\psi''\nabla w, \nabla w) + 4\tau^3((\psi''\nabla\psi \cdot \nabla\psi)w, w) - \tau((\Delta^2\psi)w, w).$$

The point is that a Carleman inequality $\|P_{0,\psi}w\| \geq c\tau^\alpha\|w\|$ for some $\alpha > 0$ may follow if the weight ψ is chosen so that the commutator term $(i[A, B]w, w)$ is at least nonnegative. In the case when ψ was a linear function, both A and B were constant coefficient operators and the commutator $i[A, B]$ was identically zero. However, the above result indicates that if ψ is for instance convex (meaning that the Hessian ψ'' is positive definite) one may obtain a better lower bound.

PROOF. The first step is to decompose $P_{0,\psi}$ into self-adjoint and skew-adjoint parts as

$$P_{0,\psi} = A + iB$$

where A and B are the formally self-adjoint operators

$$A = \frac{P_{0,\psi} + P_{0,\psi}^*}{2},$$

$$B = \frac{P_{0,\psi} - P_{0,\psi}^*}{2i}.$$

We have

$$\begin{aligned} P_{0,\psi} &= \sum_{j=1}^n (e^{\tau\psi} D_j e^{-\tau\psi})^2 = \sum_{j=1}^n (D_j + i\tau\partial_j\psi)^2 \\ &= D^2 - \tau^2|\nabla\psi|^2 + i\tau[\nabla\psi \circ D + D \circ \nabla\psi], \end{aligned}$$

and

$$P_{0,\psi}^* = (e^{\tau\psi} P_0 e^{-\tau\psi})^* = e^{-\tau\psi} P_0 e^{\tau\psi} = D^2 - \tau^2|\nabla\psi|^2 - i\tau[\nabla\psi \circ D + D \circ \nabla\psi].$$

The required expressions for A and B follow.

If $w \in H_0^2(\Omega)$ we compute

$$\|P_{0,\psi}w\|^2 = ((A + iB)w, (A + iB)w) = \|Aw\|^2 + \|Bw\|^2 + (i[A, B]w, w).$$

It remains to compute the commutator:

$$\begin{aligned}
i[A, B]w &= \tau[(D^2 - \tau^2|\nabla\psi|^2)(2\nabla\psi \cdot \nabla w + (\Delta\psi)w) \\
&\quad - (2\nabla\psi \cdot \nabla + \Delta\psi)(D^2w - \tau^2|\nabla\psi|^2w)] \\
&= \tau[2\nabla D^2\psi \cdot \nabla w + 4D\partial_k\psi \cdot D\partial_k w + (D^2\Delta\psi)w \\
&\quad + 2D\Delta\psi \cdot Dw + 2\tau^2\nabla\psi \cdot \nabla(|\nabla\psi|^2w)] \\
&= \tau[-4\nabla\Delta\psi \cdot \nabla w - 4\partial_{jk}\psi\partial_{jk}w - (\Delta^2\psi)w \\
&\quad + 4\tau^2(\psi''\nabla\psi \cdot \nabla\psi)w].
\end{aligned}$$

Integrating by parts once, using that $w|_{\partial\Omega} = 0$, yields

$$\begin{aligned}
(i[A, B]w, w) &= 4\tau(\psi''\nabla w, \nabla w) + 4\tau^3((\psi''\nabla\psi \cdot \nabla\psi)w, w) \\
&\quad - \tau((\Delta^2\psi)w, w).
\end{aligned}$$

□

PROOF OF THEOREM 3.6. In the following, the positive constants c and C will only depend on M and they may change from line to line. (We understand that c is small and C may be large.) Since $\psi = e^{\lambda\varphi}$, we have

$$\nabla\psi = \lambda e^{\lambda\varphi}\nabla\varphi, \quad \psi'' = \lambda^2 e^{\lambda\varphi}\nabla\varphi \otimes \nabla\varphi + \lambda e^{\lambda\varphi}\varphi''$$

where $\nabla\varphi \otimes \nabla\varphi$ denotes the matrix $[\partial_j\varphi\partial_k\varphi]_{j,k=1}^n$. Assuming that $\lambda \geq 1$, we also have

$$|\Delta^2\psi| \leq C\lambda^4 e^{\lambda\varphi}.$$

Let $w \in C_c^\infty(\Omega)$. By Theorem 3.7, we have

$$\|P_{0,\psi}w\|^2 = \|Aw\|^2 + \|Bw\|^2 + (i[A, B]w, w)$$

where

$$\begin{aligned}
(i[A, B]w, w) &= 4\tau(\psi''\nabla w, \nabla w) + 4\tau^3((\psi''\nabla\psi \cdot \nabla\psi)w, w) - \tau((\Delta^2\psi)w, w) \\
&= 4\tau^3\lambda^4(e^{3\lambda\varphi}|\nabla\varphi|^4w, w) + 4\tau^3\lambda^3(e^{3\lambda\varphi}(\varphi''\nabla\varphi \cdot \nabla\varphi)w, w) - \tau((\Delta^2\psi)w, w) \\
&\quad + 4\tau\lambda^2(e^{\lambda\varphi}\nabla\varphi \cdot \nabla w, \nabla\varphi \cdot \nabla w) + 4\tau\lambda(e^{\lambda\varphi}\varphi''\nabla w, \nabla w).
\end{aligned}$$

Consequently

$$\begin{aligned}
(i[A, B]w, w) &\geq 4\tau^3\lambda^3(e^{3\lambda\varphi}[\lambda|\nabla\varphi|^4 + \varphi''\nabla\varphi \cdot \nabla\varphi]w, w) - C\tau\lambda^4(e^{3\lambda\varphi}w, w) \\
&\quad - C\tau\lambda(e^{\lambda\varphi}\nabla w, \nabla w).
\end{aligned}$$

We used that $1 \leq e^{\lambda\varphi}$ and that $(e^{\lambda\varphi}\nabla\varphi \cdot \nabla w, \nabla\varphi \cdot \nabla w) \geq 0$. Now choose λ so large that $\lambda|\nabla\varphi|^4 + \varphi''\nabla\varphi \cdot \nabla\varphi \geq \lambda|\nabla\varphi|^4/2$ in $\bar{\Omega}$ (and $\lambda \geq 1$), or

$$\lambda \geq \max \left\{ 1, 2 \sup_{x \in \bar{\Omega}} \frac{\varphi''\nabla\varphi \cdot \nabla\varphi}{|\nabla\varphi|^4} \right\}.$$

This is possible since $\nabla\varphi$ is nonvanishing in $\bar{\Omega}$. If τ is chosen sufficiently large (independent of λ), it follows that

$$(i[A, B]w, w) \geq c\tau^3\lambda^4(e^{3\lambda\varphi}w, w) - C\tau\lambda(e^{\lambda\varphi}\nabla w, \nabla w).$$

We have proved the inequality

$$(3.5) \quad \|P_{0,\psi}w\|^2 \geq \|Aw\|^2 + \|Bw\|^2 + c\tau^3\lambda^4\|e^{\frac{3\lambda\varphi}{2}}w\|^2 - C\tau\lambda\|e^{\frac{\lambda\varphi}{2}}\nabla w\|^2.$$

The last negative term can be absorbed in the positive term $\|Aw\|^2$ as follows. The argument is elementary but slightly tricky. Write

$$\begin{aligned} (e^{\lambda\varphi}\nabla w, \nabla w) &= (e^{\lambda\varphi}D^2w, w) - (\nabla(e^{\lambda\varphi}) \cdot \nabla w, w) \\ &= (e^{\lambda\varphi}Aw, w) + \tau^2(e^{\lambda\varphi}|\nabla\psi|^2w, w) - \lambda(e^{\lambda\varphi}\nabla\varphi \cdot \nabla w, w) \\ &= (Aw, e^{\lambda\varphi}w) + \tau^2\lambda^2(e^{3\lambda\varphi}|\nabla\varphi|^2w, w) - \lambda(e^{\lambda\varphi}\nabla\varphi \cdot \nabla w, w). \end{aligned}$$

By Young's inequality we have $(Aw, e^{\lambda\varphi}w) \leq \frac{1}{\delta}\|Aw\|^2 + \frac{\delta}{4}\|e^{\lambda\varphi}w\|^2$ where $\delta > 0$ is a number to be determined later. We obtain

$$\|e^{\frac{\lambda\varphi}{2}}\nabla w\|^2 \leq \frac{1}{2\delta}\|Aw\|^2 + \frac{\delta}{2}\|e^{\frac{3\lambda\varphi}{2}}w\|^2 + C\tau^2\lambda^2\|e^{\frac{3\lambda\varphi}{2}}w\|^2 + C\lambda\|e^{\frac{3\lambda\varphi}{2}}w\|\|e^{\frac{\lambda\varphi}{2}}\nabla w\|$$

since $e^{\lambda\varphi} \geq 1$. Multiplying by δ and rearranging, we have

$$\begin{aligned} \frac{1}{2}\|Aw\|^2 &\geq \delta\|e^{\frac{\lambda\varphi}{2}}\nabla w\|^2 - \frac{\delta^2}{2}\|e^{\frac{3\lambda\varphi}{2}}w\|^2 - C\delta\tau^2\lambda^2\|e^{\frac{3\lambda\varphi}{2}}w\|^2 \\ (3.6) \quad &\quad - C\delta\lambda\|e^{\frac{3\lambda\varphi}{2}}w\|\|e^{\frac{\lambda\varphi}{2}}\nabla w\|. \end{aligned}$$

Writing $\|Aw\|^2 = \frac{1}{2}\|Aw\|^2 + \frac{1}{2}\|Aw\|^2$ in (3.5) and using (3.6) gives that

$$(3.7) \quad \|P_{0,\psi}w\|^2 \geq \frac{1}{2}\|Aw\|^2 + \|Bw\|^2 + R$$

where

$$\begin{aligned} R &:= \delta\|e^{\frac{\lambda\varphi}{2}}\nabla w\|^2 - \frac{\delta^2}{2}\|e^{\frac{3\lambda\varphi}{2}}w\|^2 - C\delta\tau^2\lambda^2\|e^{\frac{3\lambda\varphi}{2}}w\|^2 \\ &\quad - C\delta\lambda\|e^{\frac{3\lambda\varphi}{2}}w\|\|e^{\frac{\lambda\varphi}{2}}\nabla w\| + c\tau^3\lambda^4\|e^{\frac{3\lambda\varphi}{2}}w\|^2 - C\tau\lambda\|e^{\frac{\lambda\varphi}{2}}\nabla w\|^2. \end{aligned}$$

The idea is to choose δ so that R is positive. By inspection, we arrive at the choice

$$\delta = \varepsilon \tau \lambda^2$$

where ε is a fixed constant independent of τ and λ . If ε is chosen sufficiently small, it holds that

$$R \geq c\tau^3\lambda^4\|e^{\frac{3\lambda\varphi}{2}}w\|^2 + (\varepsilon\lambda - C)\tau\lambda\|e^{\frac{\lambda\varphi}{2}}\nabla w\|^2 - C\varepsilon\tau\lambda^3\|e^{\frac{3\lambda\varphi}{2}}w\|\|e^{\frac{\lambda\varphi}{2}}\nabla w\|.$$

Choosing λ large enough (only depending on ε and C) gives

$$R \geq c\tau^3\lambda^4\|e^{\frac{3\lambda\varphi}{2}}w\|^2 + c\tau\lambda^2\|e^{\frac{\lambda\varphi}{2}}\nabla w\|^2 - C\varepsilon\tau\lambda^3\|e^{\frac{3\lambda\varphi}{2}}w\|\|e^{\frac{\lambda\varphi}{2}}\nabla w\|.$$

Now

$$2\tau\lambda^3\|e^{\frac{3\lambda\varphi}{2}}w\|\|e^{\frac{\lambda\varphi}{2}}\nabla w\| \leq \tau^2\lambda^4\|e^{\frac{3\lambda\varphi}{2}}w\|^2 + \lambda^2\|e^{\frac{\lambda\varphi}{2}}\nabla w\|^2.$$

If τ is sufficiently large depending on C , c and ε , we have

$$(3.8) \quad R \geq c\tau^3\lambda^4\|e^{\frac{3\lambda\varphi}{2}}w\|^2 + c\tau\lambda^2\|e^{\frac{\lambda\varphi}{2}}\nabla w\|^2.$$

Going back to (3.7), the estimate (3.8) implies

$$(3.9) \quad \|P_{0,\psi}w\|^2 \geq \frac{1}{2}\|Aw\|^2 + \|Bw\|^2 + c\tau^3\lambda^4\|e^{\frac{3\lambda\varphi}{2}}w\|^2 + c\tau\lambda^2\|e^{\frac{\lambda\varphi}{2}}\nabla w\|^2.$$

The trivial estimates $\frac{1}{2}\|Aw\|^2 + \|Bw\|^2 \geq 0$ and $e^{\lambda\varphi} \geq 1$ imply that

$$\|P_{0,\psi}w\|^2 \geq c\tau^3\lambda^4\|w\|^2 + c\tau\lambda^2\|\nabla w\|^2$$

and consequently

$$\lambda^2\|w\| + \lambda\tau^{-1}\|\nabla w\| \leq C\tau^{-3/2}\|e^{\tau\psi}(-\Delta)e^{-\tau\psi}w\|.$$

Adding the potential q gives

$$\lambda^2\|w\| + \lambda\tau^{-1}\|\nabla w\|^2 \leq C\tau^{-3/2}\|e^{\tau\psi}(-\Delta + q)e^{-\tau\psi}w\| + C\tau^{-3/2}\|w\|.$$

Choosing τ so large that $C\tau^{-3/2} \leq 1/2$ and using that $\lambda \geq 1$ gives the required estimate for $w \in C_c^\infty(\Omega)$. The result for $w \in H_0^2(\Omega)$ follows by approximation. \square

3.3. Weak UCP and UCP for Cauchy data

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set, and let $q \in L^\infty(\Omega)$. We can now easily prove two other unique continuation statements:

THEOREM 3.8. (*Weak UCP*) *If $u \in H^2(\Omega)$ satisfies*

$$\begin{aligned} (-\Delta + q)u &= 0 \text{ in } \Omega, \\ u &= 0 \text{ in a ball } B \subset \Omega, \end{aligned}$$

then $u = 0$ in Ω .

THEOREM 3.9. (*UCP for local Cauchy data*) *Let $\Omega \subset \mathbb{R}^n$ have smooth boundary, and assume that Γ is a nonempty open subset of $\partial\Omega$. If $u \in H^2(\Omega)$ satisfies*

$$\begin{aligned} (-\Delta + q)u &= 0 \text{ in } \Omega, \\ u|_\Gamma &= \partial_\nu u|_\Gamma = 0, \end{aligned}$$

then $u = 0$ in Ω .

Weak unique continuation follows easily from Theorem 3.5 by using a connectedness argument. We first prove a special case.

THEOREM 3.10. (*Weak unique continuation for concentric balls*) *Let $B = B(x_0, R)$ be an open ball in \mathbb{R}^n , and let $q \in L^\infty(B)$. If $u \in H^2(B)$ satisfies*

$$(-\Delta + q)u = 0 \quad \text{in } B$$

and

$$u = 0 \text{ in some ball } B(x_0, r_0) \text{ with } r_0 < R,$$

then $u = 0$ in B .

PROOF. Let

$$I = \{r \in (0, R); u = 0 \text{ in } B(x_0, r)\}.$$

By assumption, I is nonempty. It is closed in $(0, R)$ since whenever u vanishes on $B(x_0, r_j)$ and $r_j \rightarrow r$, then u vanishes on $B(x_0, r)$. We will show that I is open, which implies $I = (0, R)$ by connectedness and therefore proves the result.

Suppose $r \in I$, so $u = 0$ in $B(x_0, r)$. Let S be the hypersurface $\partial B(x_0, r)$. We know that $u = 0$ on one side of this hypersurface. Now

Theorem 3.5 implies that for any $z \in S$, there is some open ball $B(z, r_z)$ contained in B so that u vanishes in $B(z, r_z)$. Define the open set

$$U = B(x_0, r) \cup \left(\bigcup_{z \in S} B(z, r_z) \right).$$

The distance between the compact set S and the closed set $\overline{B(x_0, R)} \setminus U$ is positive. In particular, there is some $\varepsilon > 0$ such that $u = 0$ in $B(x_0, r + \varepsilon)$. This shows that I is open. \square

PROOF OF THEOREM 3.8. Suppose $u \in H^2(\Omega)$ satisfies $(-\Delta + q)u = 0$ in Ω and $u = 0$ in some open ball contained in Ω . Set

$$A = \{x \in \Omega; u = 0 \text{ in some neighborhood of } x \text{ in } \Omega\}.$$

By assumption, A is a nonempty open subset of Ω . We will show that it is also closed. This implies by connectedness that $A = \Omega$, so indeed u vanishes in Ω as required.

Suppose on the contrary that A is not closed as a subset of Ω . Then there is some point x_0 on the boundary of A relative to Ω , for which $x_0 \notin A$. Choose $r_0 > 0$ so that $B(x_0, r_0) \subset \Omega$ and choose some point $y \in B(x_0, r_0/4)$ with $y \in A$. Since $y \in A$, we know that u vanishes on some ball $B(y, s_0)$ with $s_0 < r_0/2$. By Theorem 3.10, we see that u vanishes in the ball $B(y, r_0/2) \subset \Omega$. But $x_0 \in B(y, r_0/2)$, so u vanishes near x_0 . This contradicts the fact that $x_0 \notin A$. \square

In turn, unique continuation from Cauchy data on a subset follows from weak unique continuation upon extending the domain slightly near the set where the Cauchy data vanishes.

PROOF OF THEOREM 3.9. Assume that $u \in H^2(\Omega)$, $(-\Delta + q)u = 0$ in Ω , and $u|_\Gamma = \partial_\nu u|_\Gamma = 0$. Choose some $x_0 \in \Gamma$, and choose coordinates $x = (x', x_n)$ so that $x_0 = 0$ and for some $r > 0$,

$$\Omega \cap B(0, r) = \{x \in B(0, r); x_n > g(x')\}$$

where $g : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ is a C^∞ function. We extend the domain near x_0 by choosing $\psi \in C_c^\infty(\mathbb{R}^{n-1})$ with $\psi = 0$ for $|x'| \geq r/2$ and $\psi = 1$ for $|x'| \leq r/4$, and by letting

$$\tilde{\Omega} = \Omega \cup \{x \in B(0, r); x_n > g(x') - \varepsilon\psi(x')\}.$$

Here $\varepsilon > 0$ is chosen so small that $\{(x', x_n); |x'| \leq r/2, x_n = g(x') - \varepsilon\psi(x')\}$ is contained in $B(0, r)$. Then $\tilde{\Omega}$ is a bounded connected open set with smooth boundary.

Define the function

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \tilde{\Omega} \setminus \Omega. \end{cases}$$

Then $\tilde{u}|_{\Omega} \in H^2(\Omega)$ and $\tilde{u}|_{\tilde{\Omega} \setminus \Omega} \in H^2(\tilde{\Omega} \setminus \Omega)$. Since $u|_{\Gamma} = \partial_{\nu} u|_{\Gamma} = 0$, we also have that the traces of \tilde{u} and $\partial_{\nu} \tilde{u}$ on the interface $\partial\Omega \setminus \partial\tilde{\Omega}$ vanish when taken both from inside and outside Ω .

It follows that $\tilde{u} \in H^2(\tilde{\Omega})$. Defining $\tilde{q}(x) = q(x)$ for $x \in \Omega$ and $\tilde{q}(x) = 0$ for $\tilde{\Omega} \setminus \Omega$, one also gets that $(-\Delta + \tilde{q})\tilde{u} = 0$ almost everywhere in $\tilde{\Omega}$. But $\tilde{u} = 0$ in some open ball contained in $\tilde{\Omega} \setminus \overline{\Omega}$, so we know from Theorem 3.8 that $\tilde{u} = 0$ in the connected domain $\tilde{\Omega}$. Thus also $u = 0$. \square

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