Integral Equations and Boundary Value Problems

Exercises

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1. Classify the following integral equations and identify, if possible, the kernel in each equation:

   (a) $\int_0^s (-e^{i\pi} + \beta)x(t)\,dt = (\beta + 1)x(s)\,,$

   (b) $g(x) = \int_{-t}^{t} (x - y)^2 \sqrt{f(y)}\,dy + s(x)g(x)\,,$

   (c) $\alpha f(s) - \int_u^v 2\sin(\pi x)f(x)\,dx = \cos(2\pi s)f(s)\,.$

2. Rewrite the following boundary value problem using an integral equation and classify it:

$$y''(s) + \lambda y(s) = 0,\ \ y(0) = y(1) = 0,\ \ s \in [0, 1]\,.$$

3. Prove or contradict: The Fredholm integral equation

$$x(s) - \int_0^1 x(t)\,dt = s\,,$$

   is solvable in $C[0, 1]$.

4. Please show that the integral equation

$$g(s) = f(s) + \frac{1}{\pi}\int_0^{2\pi} \sin(s + t)g(t)\,dt\,,$$

   has no solution for $f(s) = s$. What can be said about the case $f(s) = 1$?
   Hint: Show and use that the kernel is degenerate.

5. Show that the integral operator $K : C[0, 1] \to C[0, 1]$ induced by the singular kernel

$$k(s, t) = \log|s - t|\,, \qquad s, t \in [0, 1], s \neq t\,,$$

   is compact.

6. Solve the following integral equation:

$$g(s) - \lambda \int_0^1 (20st^2 + 12s^2t)g(t)\,dt = s\,, \qquad s \in [0, 1]\,.$$

7. Let K be the integral operator induced by the kernel $k(s,t) := e^{-t^2(s+1)}$ on $C[0,1]$ with the supremum-norm. Calculate the Riesz index of $L := I - K$. Consider the integral equation

$$x(s) - \int_0^1 k(s,t)x(t)\,dt = f_i(s)\,, \quad s \in [0,1]\,,\ i = 1,2,3\,,$$

with $f_1(x) = \sin s,\ f_2(x) = s^3 - e^s$ and $f_3(s) = e^{s^2}$. For which of the right hand sides is the integral equation solvable?

8. Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual system. Let $S : X \to X$ and $T : Y \to Y$ be linear continuously invertible operators on X and Y respectively and let $A : X \to X$ and $B : Y \to Y$ be compact operators, such that S is adjoint to T and A is adjoint to B. Please show that:

   (a) the homogeneous equations
   $$Sx - Ax = 0\,,$$
   and
   $$Ty - By = 0\,,$$
   have the same number of linearly independent solutions.

   (b) the inhomogeneous equation
   $$Sx - Ax = f, \qquad f \in X\,,$$
   has a solution if and only if for all solutions y of $Ty - By = 0$, $\langle f, y\rangle = 0$.

9. Let $K : L^2[0,1] \to L^2[0,1]$ be the integral operator induced by the kernel

$$k(s,t) := 4\pi^2 \begin{cases} (1-s)t\,, & t \le s\,, \\ (1-t)s\,, & t > s\,. \end{cases}$$

Compute $\mathcal{N}(I-K)$ and $\mathcal{N}(I-K')$.

10. Let $\langle X, Y\rangle$ be a dual system with two normed spaces, X and Y. Let $A_n : X \to X$ and $B_n : Y \to Y$ be adjoint finite dimensional operators of the form,

$$A_n\phi = \sum_{j=1}^{n} \langle \phi, b_j\rangle\, a_j \quad \text{and} \quad B_n\psi = \sum_{j=1}^{n} \langle a_j, \psi\rangle\, b_j\,,$$

for $\phi \in X$ and $\psi \in Y$, and linear independent elements $\{a_1, \ldots, a_n\} \subset X$ and $\{b_1, \ldots, b_n\} \subset Y$. By reducing the operator equations to linear systems, demonstrate the validity of Fredholm's theorem for the operators $I - A_n$ and $I - B_n$.