Integration of differential forms

There is a close relation between anti-symmetric tensors (differential forms) and volume. For instance, for the n-form $dx^1 \wedge \ldots \wedge dx^n$ acting on vectors $v_1, \ldots, v_n$ we have

$$dx^1 \wedge \ldots \wedge dx^n (v_1, \ldots, v_n) = \sum_{i_1 \ldots i_n}^{n} \epsilon_{i_1 \ldots i_n} dx^{i_1} \otimes \ldots \otimes dx^{i_n} (v_1, \ldots, v_n)$$

$$= \sum_{i_1 \ldots i_n}^{n} \epsilon_{i_1 \ldots i_n} v_1^{i_1} \ldots v_n^{i_n} = \det(v_1, \ldots, v_n)$$

$$= \int_{\mathrm{Vol}(v_1, \ldots v_n)} dx^1 \ldots dx^n \qquad (1)$$

where $\mathrm{vol}(v_1, \ldots, v_n)$ is the volume spanned by the vectors $(v_1, \ldots, v_n)$. Eq. (1) shows that there is a close relation between differential forms and integration.

In fact, a very important property of differential forms is that they can be integrated over regions of space (manifolds) independent of the coordinate system. There are many instances in physics where we need to do this, for example by carrying line integrals along force fields, flux integrals over surfaces, or volume integrals over Lagrangians. Let us start with some simple examples. The simplest manifold we can think of is simply a 1-dimensional interval $[y_1, y_2]$ on which we can define the 1-form

$$\omega = f(y)\, dy \qquad (2)$$

where we use the coordinate $y_1 \leq y \leq y_2$. We then define

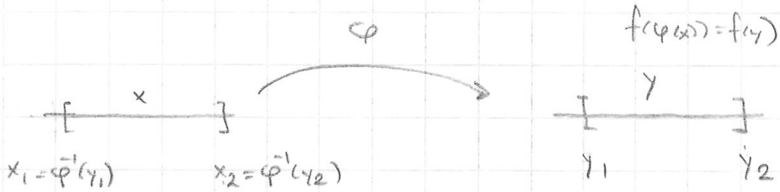
$$\int_{[y_1, y_2]} \omega \equiv \int_{y_1}^{y_2} f(y)\, dy \qquad (3)$$

By the substitution rule this integral is independent of the parametrization. If we have

$$y = \varphi(x) \qquad (4)$$

for some smooth function φ then

$$\int_{y_1}^{y_2} f(y)\,dy = \int_{x_1}^{x_2} f(\varphi(x))\,\frac{d\varphi}{dx}\,dx \qquad (5)$$



We also have that

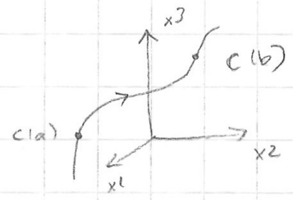
$$\varphi^* \omega = \varphi^*(f(y)\,dy) = f(\varphi(x))\,d\varphi(x) = f(\varphi(x))\,\frac{d\varphi}{dx}\,dx \qquad (6)$$

is the pullback of differential form $\omega = f(y)dy$ under the mapping φ. So Eq (5) can also be written as

$$\int_{\varphi[x_1,x_2]} \omega = \int_{[x_1,x_2]} \varphi^* \omega \qquad (7)$$

i.e. if ω is an integral over an interval $[y_1,y_2] = \varphi[x_1,x_2]$ we can pull it back by φ to the interval $[x_1,x_2]$ and integrate $\varphi^*\omega$ over this interval.
This simple idea has the following useful application
Let $c(t) = (c^1(t), \dots, c^n(t))$ be a curve in an n-dimensional space

$$c : [a,b] \to \mathbb{R}^n \qquad (8)$$



and

$$F = \sum_{i=1}^{n} F_i(x)\,dx^i \qquad (9)$$

be a one-form then we can define the line-integral of F along c to be

$$\int_{c[a,b]} F \equiv \int_{[a,b]} c^* F \qquad (10)$$

i.e. we pull back F to a one-form c^*F on the interval $[a,b]$ and integrate over the interval.
Let us work this out more explicitly

The 1-form F has the form

$$F = \sum_{j=1}^{n} F_j(x) dx^j \tag{11}$$

Then

$$\begin{aligned} c^*F &= \sum_{j=1}^{n} F_j(c(t))\, dc^j(t) \\ &= \sum_{j=1}^{n} F_j(c(t)) \frac{dc^j}{dt} dt = \langle F, \frac{dc}{dt} \rangle dt \end{aligned} \tag{12}$$

where we defined

$$\langle F, G \rangle = \sum_{j=1}^{n} F_j G^j \tag{13}$$

for a vector G. Eq. (10) therefore becomes

$$\int_{c[a,b]} F = \int_a^b dt \langle F, \frac{dc}{dt} \rangle \tag{14}$$

which is the familiar line integral of a vector field F along a curve $c(t)$

It is clear that we can generalize this idea. The generalization of integrating over an interval $[a,b]$ it the integral over a p-dimensional cube $I^p = [a_1, b_1] \times \dots \times [a_p, b_p]$, $a_j < b_j$. If

$$\omega = f(t^1, \dots, t^p)\, dt^1 \wedge \dots \wedge dt^p \tag{15}$$

Then

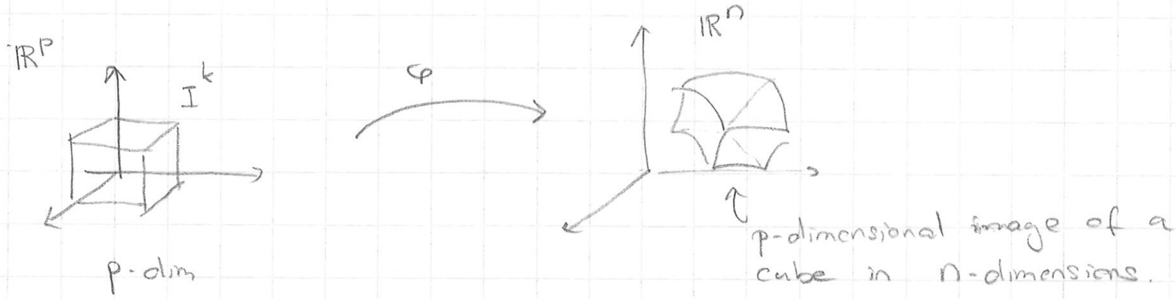
$$\int_{I^p} \omega \equiv \int_{a_1}^{b_1} dt^1 \dots \int_{a_p}^{b_p} dt^p\, f(t^1, \dots, t^p) \tag{16}$$

The generalization of the curve $c: I \to \mathbb{R}^n$ is now a map $\varphi: I^p \to \mathbb{R}^n$ $(n \geq p)$ and we define for a p-form

$$\omega = \sum_{i_1 < \dots < i_p}^{n} \omega_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} \tag{17}$$

on $\mathbb{R}^n$ that

$$\int_{\varphi[I^p]} \omega = \int_{I^p} \varphi^* \omega \tag{18}$$



(19)

Let us work this out in more detail. The map φ is of the form

$$\varphi = (\varphi^1(t^1,\ldots,t^p),\ldots,\varphi^n(t^1,\ldots,t^p)) \tag{20}$$

Then $(t = (t^1,\ldots,t^p))$

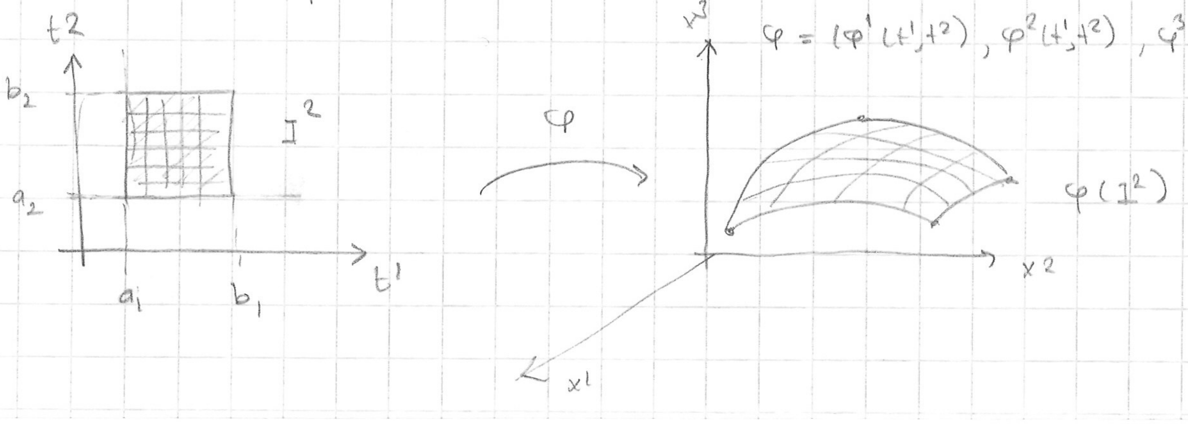
$$\begin{aligned}
\varphi^*\omega &= \sum_{i_1<\ldots<i_p}^{n} \omega_{i_1\ldots i_p}(\varphi(t))\, d\varphi^{i_1}\wedge\ldots\wedge d\varphi^{i_p} \\
&= \sum_{i_1<\ldots<i_p}^{n} \sum_{j_1\ldots j_p}^{p} \omega_{i_1\ldots i_p}(\varphi(t)) \frac{d\varphi^{i_1}}{dt^{j_1}}\cdots\frac{d\varphi^{i_p}}{dt^{j_p}}\, dt^{j_1}\wedge\ldots\wedge dt^{j_p} \\
&= \sum_{i_1<\ldots<i_p}^{n} \sum_{j_1\ldots j_p}^{p} \omega_{i_1\ldots i_p}(\varphi(t)) \frac{d\varphi^{i_1}}{dt^{j_1}}\cdots\frac{d\varphi^{i_p}}{dt^{j_p}}\, \epsilon_{j_1\ldots j_p}\, dt^1\wedge\ldots\wedge dt^p \\
&= \sum_{i_1<\ldots<i_p}^{n} \omega_{i_1\ldots i_p}(\varphi(t)) \begin{vmatrix} \frac{\partial\varphi^{i_1}}{\partial t^1} & \cdots & \frac{\partial\varphi^{i_1}}{\partial t^p} \\ \vdots & & \vdots \\ \frac{\partial\varphi^{i_p}}{\partial t^1} & \cdots & \frac{\partial\varphi^{i_p}}{\partial t^p} \end{vmatrix} dt^1\wedge\ldots\wedge dt^p
\end{aligned} \tag{21}$$

and we thus find that

$$\int_{\varphi[I^p]} \omega = \sum_{i_1<\ldots<i_p}^{n} \int_{a_1}^{b_1} dt^1 \cdots \int_{a_p}^{b_p} dt^p\, \omega_{i_1\ldots i_p}(\varphi(t)) \begin{vmatrix} \frac{\partial\varphi^{i_1}}{\partial t^1} & \cdots & \frac{\partial\varphi^{i_1}}{\partial t^p} \\ \vdots & & \vdots \\ \frac{\partial\varphi^{i_p}}{\partial t^1} & \cdots & \frac{\partial\varphi^{i_p}}{\partial t^p} \end{vmatrix} \tag{22}$$

A well-known special case of this integral is the case that $p=2$ and $n=3$

$$\varphi = (\varphi^1(t^1,t^2), \varphi^2(t^1,t^2), \varphi^3(t^1,t^2))$$



(23)

In this case I^2 is a square in the (t^1, t^2)-plane and $\varphi(I^2)$ is a 2-dimensional surface in 3 dimensions. Writing out (22) explicitly for this case gives

$$\int_{\varphi(I^2)} \omega = \int_{a_1}^{b_1} dt^1 \int_{a_2}^{b_2} dt^2 \left\{ \omega_{12}(\varphi) \begin{vmatrix} \frac{\partial\varphi^1}{\partial t^1} & \frac{\partial\varphi^1}{\partial t^2} \\ \frac{\partial\varphi^2}{\partial t^1} & \frac{\partial\varphi^2}{\partial t^2} \end{vmatrix} \right.$$

$$\left. + \omega_{13}(\varphi) \begin{vmatrix} \frac{\partial\varphi^1}{\partial t^1} & \frac{\partial\varphi^1}{\partial t^2} \\ \frac{\partial\varphi^3}{\partial t^1} & \frac{\partial\varphi^3}{\partial t^2} \end{vmatrix} + \omega_{23}(\varphi) \begin{vmatrix} \frac{\partial\varphi^2}{\partial t^1} & \frac{\partial\varphi^2}{\partial t^2} \\ \frac{\partial\varphi^3}{\partial t^1} & \frac{\partial\varphi^3}{\partial t^2} \end{vmatrix} \right\} \qquad (24)$$

If we denote $\omega_{12} = b_3$, $\omega_{13} = -b$, $\omega_{23} = b_1$, or $b_i = \epsilon_{ijk}\omega_{jk}$ then we can write this equation as

$$\int_{\varphi(I^2)} b_1\, dx^2 \wedge dx^3 + b_2\, dx^3 \wedge dx^2 + b_3\, dx^1 \wedge dx^2$$

$$= \int_{a_1}^{b_1} dt^1 \int_{a_2}^{b_2} dt^2 \left\{ b_3(\varphi(t)) \begin{vmatrix} \frac{\partial\varphi^1}{\partial t^1} & \frac{\partial\varphi^1}{\partial t^2} \\ \frac{\partial\varphi^2}{\partial t^1} & \frac{\partial\varphi^2}{\partial t^2} \end{vmatrix} - b_2(\varphi(t)) \begin{vmatrix} \frac{\partial\varphi^1}{\partial t^1} & \frac{\partial\varphi^1}{\partial t^2} \\ \frac{\partial\varphi^3}{\partial t^1} & \frac{\partial\varphi^3}{\partial t^2} \end{vmatrix} \right.$$

$$\left. + b_1(\varphi(t)) \begin{vmatrix} \frac{\partial\varphi^2}{\partial t^1} & \frac{\partial\varphi^2}{\partial t^2} \\ \frac{\partial\varphi^3}{\partial t^1} & \frac{\partial\varphi^3}{\partial t^2} \end{vmatrix} \right\}$$

$$= \int_{a_1}^{b_1} dt^1 \int_{a_2}^{b_2} dt^2 \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial\varphi^1}{\partial t^1} \\ \frac{\partial\varphi^2}{\partial t^1} \\ \frac{\partial\varphi^3}{\partial t^1} \end{pmatrix} \times \begin{pmatrix} \frac{\partial\varphi^1}{\partial t^2} \\ \frac{\partial\varphi^2}{\partial t^2} \\ \frac{\partial\varphi^3}{\partial t^2} \end{pmatrix}$$

$$= \int_{a_1}^{b_1} dt^1 \int_{a_2}^{b_2} dt^2 \; \vec{b} \cdot \left(\frac{\partial\varphi}{\partial t^1} \times \frac{\partial\varphi}{\partial t^2} \right)$$

$$= \int dA \; \vec{b} \cdot \vec{n} \qquad (25)$$

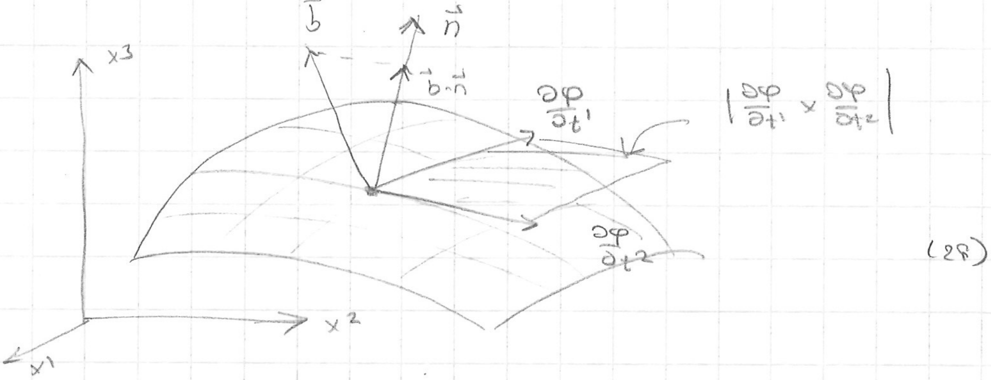
where

$$\vec{n} = \frac{\frac{\partial\varphi}{\partial t^1} \times \frac{\partial\varphi}{\partial t^2}}{\left| \frac{\partial\varphi}{\partial t^1} \times \frac{\partial\varphi}{\partial t^2} \right|} \qquad (26)$$

and

$$dA = \left| \frac{\partial\varphi}{\partial t^1} \times \frac{\partial\varphi}{\partial t^2} \right| dt^1 dt^2 \qquad (27)$$

Here $\vec{n}$ is a normal vector to the surface, $\vec{b}\cdot\vec{n}$ the Euclidean inner product and dA the surface area element



(28)

Eq. (25) represents the integral for the flux of a vector field $\vec{b}$ through surface $\varphi(I^2)$.

After these preliminaries we now address a central idea in the integration of differential forms. If ω is a p-form then $d\omega$ is a $(p+1)$-form. Since the d-operation is a kind of differentiation we may think to get back to ω by integration. Let us try this idea for the $(n-1)$-form

$$\omega = \sum_{j=1}^{n} \omega_j(x)\, dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^n \tag{29}$$

in an n-dimensional space. Here $\widehat{dx^j}$ denotes that dt^j is missing from the wedge product. Then

$$\begin{aligned} d\omega &= \sum_{j=1}^{n} \frac{\partial\omega_j}{\partial x^j}\, dx^j \wedge dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^n \\ &= \sum_{j=1}^{n} \frac{\partial\omega_j}{\partial x^j} (-1)^{j-1}\, dx^1 \wedge \dots \wedge dx^n \end{aligned} \tag{30}$$

This is an n-form that we can integrate over the cube $I^n = [a_1,b_1]\times\dots\times[a_n,b_n]$. We have

$$\begin{aligned} \int_{I^n} d\omega &= \sum_{j=1}^{n} \int_{a_1}^{b_1} dx^1 \dots \int_{a_n}^{b_n} dx^n\, \frac{\partial\omega_j}{\partial x^j} (-1)^{j-1} \\ &= \sum_{j=1}^{n} \int_{a_1}^{b_1} dx^1 \dots \widehat{dx^j} \dots \int_{a_n}^{b_n} dx^n\, (-1)^{j-1} \Big[\omega_j(x^1, \dots, x^{j-1}, b_j, x^{j+1}, \dots, x^n) \\ &\qquad - \omega_j(x^1, \dots, x^{j-1}, a_j, x^{j+1}, \dots, x^n) \Big] \end{aligned}$$

$$= \sum_{\alpha=0}^{1} \sum_{j=1}^{n} (-1)^{j+\alpha} \int_{a_1}^{b_1} dx^1 \ldots \widehat{dx^j} \ldots \int_{a_n}^{b_n} dx^n \; \omega_j(x^1, \ldots, x^{j-1}, a_j + \alpha(b_j - a_j), x^{j+1}, \ldots, x^n) \tag{31}$$

We would now like to interpret the last term in (31) as the integral of ω over the edges

$$I^n_{(j,\alpha)} = (x^1, \ldots, x^{j-1}, a_j + \alpha(b_j - a_j), x^{j+1}, \ldots, x^n) \tag{32}$$

of the cube. To do this we have to address the signs $(-1)^{j+\alpha}$ in (31). Let us how this work out in a special case. For simplicity we take $[a_j, b_j] = [0, 1]$, $j = 1, \ldots n$, and let $n = 2$. Then (29) and (30) tell us that if

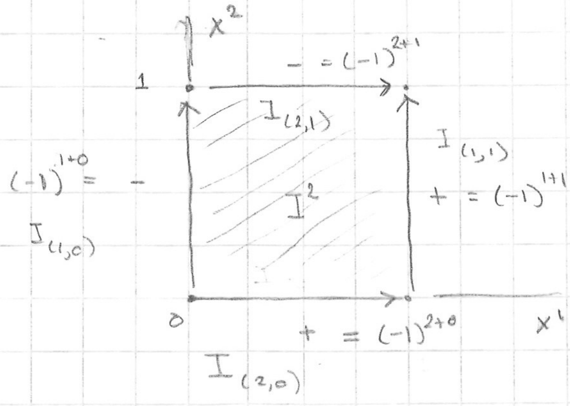
$$\omega = \omega_1(x)\, dx^2 + \omega_2(x)\, dx^1 \tag{33}$$

then $\quad d\omega = \left(\frac{\partial \omega_1}{\partial x^1} - \frac{\partial \omega_2}{\partial x^2} \right) dx^1 \wedge dx^2 \quad$ and

$$\int_{I^2} d\omega = \int_0^1 dx^1 \int_0^1 dx^2 \left(\frac{\partial \omega_1}{\partial x^1} - \frac{\partial \omega_2}{\partial x^2} \right)$$

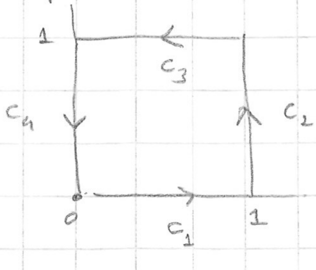
$$= \int_0^1 dx^1 \; \omega_2(x^1, 0) + \int_0^1 dx^2 \; \omega_1(1, x^2) - \int_0^1 dx^1 \; \omega_2(x^1, 1) - \int_0^1 dx^2 \; \omega_1(0, x^2) \tag{34}$$

The last 4 terms in (34) amounts to integration along the side of a square with the following signs



(35)

Let us compare (34) to the line integral of (33) along



$$C = C_1 + C_2 + C_3 + C_4 \tag{36}$$

curve C which forms the boundary of I^2.

This integral is the sum of 4 pieces

$$\int_c \omega = \int_{c_1} \omega + \int_{c_2} \omega + \int_{c_3} \omega + \int_{c_4} \omega \qquad (37)$$

Each of these pieces can be calculated separately as follows. For $c_1(t)$ we can take the parametrization

$$c_1(t) = (c_1^1(t), c_1^2(t)), = (t, 0) \quad , \quad t \in [0,1] \qquad (38)$$

and

$$\int_{c_1} \omega = \int_0^1 dt \left[\omega_1(t,0) \frac{dc_1^2}{dt} + \omega_2(t,0) \frac{dc_1^1}{dt} \right]$$

$$= \int_0^1 dt \ \omega_2(t,0) \qquad (39)$$

Similarly for $c_2(t) = (1, t)$, $t \in [0,1]$

$$\int_{c_2} \omega = \int_0^1 dt \ \omega_1(1,t) \qquad (40)$$

For c_3 we have $c_3(t) = (1-t, 1)$ $t \in [0,1]$ and $\frac{dc_3}{dt} = (-1, 0)$. Then

$$\int_{c_3} \omega = - \int_0^1 dt \ \omega_2(1-t, 1) \overset{s = 1-t}{=} \int_1^0 ds \ \omega_2(s,1)$$

$$= - \int_0^1 ds \ \omega_2(s,1) \qquad (41)$$

and similarly for $c_4(t) = (0, 1-t)$ $t \in [0,1]$
$\frac{dc_4}{dt} = (0, -1)$

$$\int_{c_4} \omega = - \int_0^1 dt \ \omega_1(0, 1-t) \overset{s=1-t}{=} \int_1^0 ds \ \omega_1(0,s)$$

$$= - \int_0^1 ds \ \omega_1(0,s) \qquad (42)$$

Adding (39) - (42) then gives

$$\int_c \omega = \int_0^1 ds \ \omega_2(s,0) + \int_0^1 ds \ \omega_1(1,s) - \int_0^1 ds \ \omega_2(s,1) - \int_0^1 ds \ \omega_1(0,s) \qquad (43)$$

which is identical to the right hand side of (34). We thus see that

$$\int_{I^2} d\omega = \int_{\partial I^2} \omega \qquad (44)$$

where ∂I^2 denotes the integral along the edge of the square with the orientation of the loop chosen as in (36).

We can expect that a relation as (45) applies to general n-dimensional cubes, i.e.

$$\int_{I^n} d\omega = \int_{\partial I^n} \omega \qquad (45)$$

, however, our derivation shows that we have to choose the orientation of the boundary integral carefully. To discuss the general case we first need to define what we mean with choosing an orientation.

If $v_1, \dots, v_n$ are n vectors in an n-dimensional space we say that the basis is positively oriented whenever

$$\det(v_1, \dots, v_n) > 0 \qquad (46).$$

Conversely if $\det(v_1, \dots, v_n) < 0$ we say that the basis is negatively oriented. In a coordinate basis we can write every vector as

$$v_k = \sum_{j=1}^{n} v_k^j \frac{\partial}{\partial x^j} \qquad (47)$$

and

$$\det(v_1, \dots, v_n) = dx^1 \wedge \dots \wedge dx^n (v_1, \dots, v_n) \qquad (48)$$

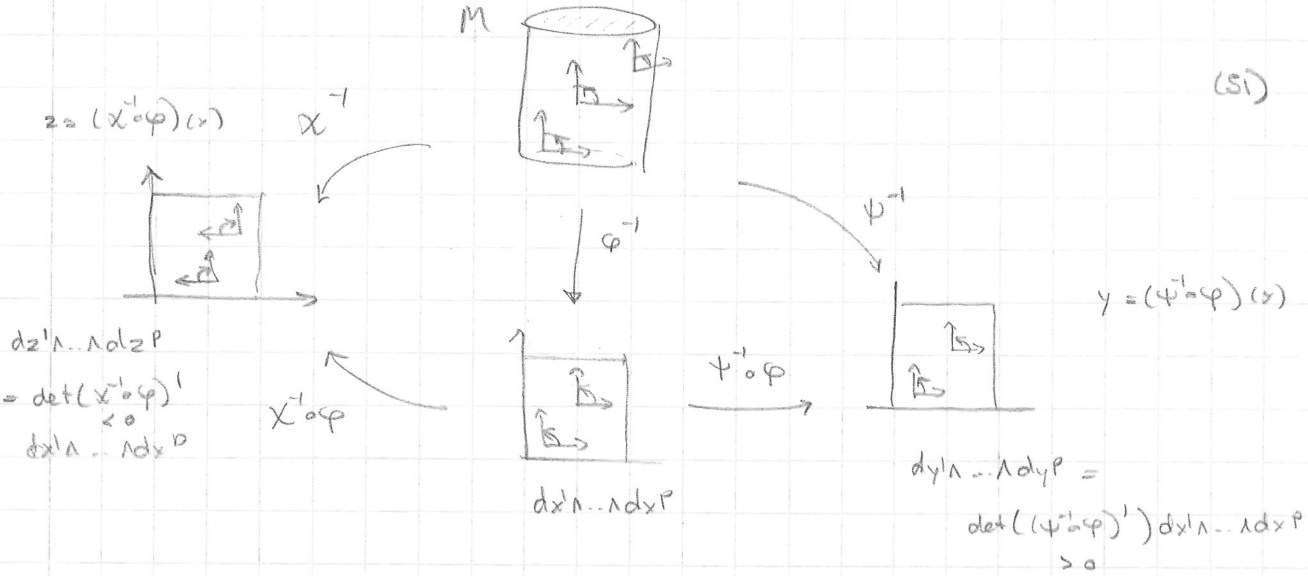
More generally a manifold M is said to be orientable when there exists a nowhere vanishing n-form Ω, i.e.

$$\Omega_p(e_1, \dots, e_n) \neq 0 \qquad \forall\, p \in M \qquad (49)$$

where $e_i = \frac{\partial}{\partial x^i}$. Clearly the n-form

$$\Omega = dx^1 \wedge \dots \wedge dx^n \qquad (50)$$

satisfies this criterium on a Euclidean space. A classic example of a non-orientable manifold is the Möbius strip. However, in the following we will only consider orientable manifolds. On such a manifold we can have different coordinate maps



(51)

If $y = (\psi^{-1} \circ \varphi)(x)$ for two coordinate maps φ^{-1} and ψ^{-1} we say that $\psi^{-1} \circ \varphi$ is orientation preserving when $\det(\psi^{-1} \circ \varphi)' > 0$ and orientation reversing when $\det(\psi^{-1} \circ \varphi)' < 0$, where $\det(\psi^{-1} \circ \varphi)'$ is the Jacobian of the transformation $\psi^{-1} \circ \varphi$. On an orientable manifold there exists an orientation preserving atlas of coordinate transformations.

For a set of vectors $v_1, \ldots, v_n$ let us define

$$\mu = \text{sign}\left[dx^1 \wedge .. \wedge dx^n (v_1, \ldots, v_n) \right]$$

$$\equiv \left[v_1, \ldots, v_n \right] \tag{52}$$

For a positively orientated basis we have

$$\left[v_1, \ldots, v_n \right] = 1 \tag{53}$$

whereas for a negatively oriented basis we have

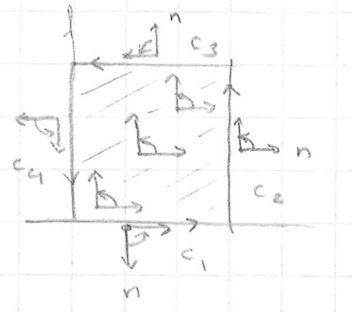
$$\left[v_1, \ldots, v_n \right] = -1 \tag{54}$$

Let $w_j = A v_j$ where A is a $n \times n$ - matrix.

Then

$$dx^1 \wedge \dots \wedge dx^n \, (w_1, \dots, w_n) = (\det A) \; dx^1 \wedge \dots dx^n \, (v_1, \dots, v_n) \tag{55}$$

Then every transformation with $\det A > 0$ preserves the orientation while the orientation is reversed when $\det A < 0$. For instance, in Euclidean space rotations preserve the orientation while mirrorings reverse them. Let us now go back to our example of the square of (36).



(56)

Inside the square $dx^1 \wedge dx^2$ is a nonvanishing 2-form an therefore the square is orientable. The basis vectors

$$e_1 = \frac{\partial}{\partial x^1} \quad , \quad e_2 = \frac{\partial}{\partial x^2} \tag{57}$$

form a positively orientated basis

$$[e_1, e_2] = 1 \tag{58}$$

since

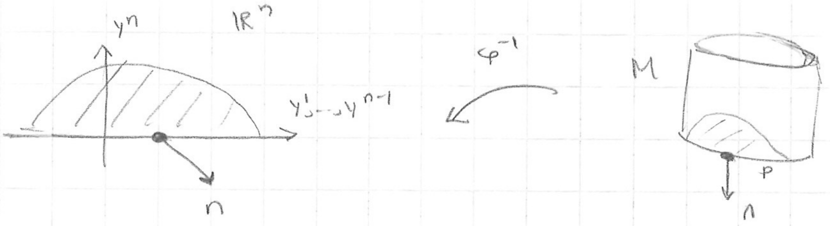
$$dx^1 \wedge dx^2 \; (e_1, e_2) = 1 > 0 \; . \tag{59}$$

We see from (56) that we get the wanted orientation along the edge ∂I^2 if we require that

$$[n, v] = 1 \tag{60}$$

where n is an outward pointing vector and $v \in \partial I^2$. The vector n is called the outword normal vector although it is in fact not required that it is normalized (no metric at all needs to be introduced).

For a general manifold M we can always find a coordinate system φ such that for a point p on the boundary $\varphi^{-1}(p) = (y^1, \dots, y^{n-1}, 0)$ whereas for points close to the boundary $y^n \geq 0$. Pictorially:



(61)

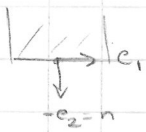
An outward normal is then of the form

$$v = \sum_{j=1}^{n} v^j(p) \frac{\partial}{\partial y^j} \qquad (62)$$

with $v^n(p) < 0$. If $v^n(p) = 0$ then v is a tangent vector of the boundary.

Let us now go back to (56). For the edge c_1 of the boundary we have

$$n = -e_2$$



and we need to choose $v = e_1 \in \partial I^2$ to satisfy (60), i.e.

$$[-e_2, e_1] = [e_1, e_2] = 1 \qquad (63)$$

Since $\frac{dc_1}{dt} = (1, 0) = \frac{\partial}{\partial x^1}$ (or more formally $c_{1*}\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial x^1}$)

we see that $\frac{\partial}{\partial x^1} = e_1$ corresponds to the orientation chosen in (36).

For the remaining edges c_2, c_3, c_3 we have similarly

c_2: $\quad n = e_1, \quad v = e_2 = \frac{\partial}{\partial x^2} = (0, 1) = \frac{dc_2}{dt}$

$$[n, v] = [e_1, e_2] = 1 \qquad (64)$$

c_3: $\quad n = e_2, \quad v = -e_1 = -\frac{\partial}{\partial x^1} = (0, -1) = \frac{dc_3}{dt}$

$$[n, v] = [e_2, -e_1] = 1 \qquad (65)$$

c_4: $\quad n = -e_1, \quad v = -e_2 = -\frac{\partial}{\partial x^2} = (-1, 0) = \frac{dc_4}{dt}$

$$[n, v] = [-e_1, -e_2] = 1 \qquad (66)$$

We are now ready to extend our discussion to the case of an p-dimensional cube. Let n be an outward normal. Then we choose vectors $v_1, \dots, v_{p-1} \in \partial I^p$ such that

$$[n, v_1, \dots, v_{p-1}] = 1 \tag{67}$$

which means that

$$dx^1 \wedge \dots \wedge dx^p \, (n, v_1, \dots, v_{p-1}) > 0 \tag{68}$$

Let us first consider the side $I_{(\alpha,j)}$, i.e. the side with coordinate $x^j = \alpha = 0, 1$ $([a_k, b_k] = [0,1])$.
Then

$$n = \frac{\partial}{\partial x^j} \quad (\alpha = 1) \quad , \quad n = -\frac{\partial}{\partial x^j} \quad (\alpha = 0) \tag{69}$$

and therefore

$$dx^1 \wedge \dots \wedge dx^p \, (n, v_1, \dots, v_{p-1})$$

$$= (-1)^{j-1} dx^j \wedge dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots dx^p \, (\pm \frac{\partial}{\partial x^j}, v_1, \dots v_{p-1})$$

$$= (-1)^{j+\alpha} dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^p \, (v_1, \dots, v_{p-1}) > 0 \tag{70}$$

Let now φ be a parametrization of edge $I_{(\alpha,j)}$, i.e.

$$\varphi = (\varphi^1(t^1, \dots, t^{p-1}), \dots, \varphi^{j-1}(t^1, \dots, t^{p-1}), \alpha, \varphi^{j+1}(t^1, \dots, t^{p-1}), \dots, \varphi^p(t^1, \dots, t^{p-1})) \tag{71}$$

and let

$$v^k = \varphi_* \left(\frac{\partial}{\partial t^k} \right) \tag{72}$$

be the push-forward of the coordinate vector $\frac{\partial}{\partial t^k}$ by φ. Then (70) tells us that

$$(-1)^{j+\alpha} dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^p \left(\varphi_* \left(\frac{\partial}{\partial t^1} \right), \dots, \varphi_* \left(\frac{\partial}{\partial t^{p-1}} \right) \right)$$

$$= (-1)^{j+\alpha} \varphi^* (dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^p) \left(\frac{\partial}{\partial t^1}, \dots, \frac{\partial}{\partial t^{p-1}} \right)$$

$$= (-1)^{j+\alpha} d\varphi^1 \wedge \dots \wedge \widehat{d\varphi^j} \wedge \dots \wedge d\varphi^p \left(\frac{\partial}{\partial t^1}, \dots, \frac{\partial}{\partial t^{p-1}} \right) > 0 \tag{73}$$

For a proper orientation we thus have to choose our parametrization such that (73) is satisfied. For $j+q$ is even this is simplify achieved by

$$\varphi = (t^1, \dots, t^{j-1}, q, t^j, \dots, t^{p-1}) \tag{74}$$

and
$$d\varphi^1 \wedge \dots \wedge \widehat{d\varphi^j} \wedge \dots \wedge d\varphi^p = dt^1 \wedge \dots \wedge dt^{p-1} \tag{75}$$

For $j+q$ odd we can simply interchange two variables in (74). For example, if $j=3$, then we can interchange t^1 and t^2 and use

$$\varphi = (t^2, t^1, q, t^3, \dots, t^{p-1}) \tag{76}$$

and
$$\begin{aligned} d\varphi^1 \wedge \dots \wedge \widehat{d\varphi^j} \wedge \dots \wedge d\varphi^p &= dt^2 \wedge dt^1 \wedge dt^3 \wedge \dots \wedge dt^{p-1} \\ &= - dt^1 \wedge dt^2 \wedge \dots \wedge dt^{p-1} \end{aligned} \tag{77}$$

In both cases (75) and (77) we then have

$$d\varphi^1 \wedge \dots \wedge \widehat{d\varphi^j} \wedge \dots \wedge d\varphi^p = (-1)^{j+q} dt^1 \wedge \dots \wedge dt^{p-1} \tag{78}$$

and therefore

$$\int_{I(q,j)} \omega_j(x)\, dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^p$$

$$= \int_0^1 dt^1 \dots \int_0^1 dt^{p-1}\, \omega_j(\varphi(t))\, (-1)^{j+q}$$

$$= (-1)^{j+q} \int_0^1 ds^1 \dots \widehat{ds^j} \dots \int_0^1 ds^p\, \omega_j(s^1, \dots, s^{j-1}, q, s^{j+1} \dots s^p) \tag{79}$$

where in the last step we relabeled $(t^1, \dots, t^{p-1}) = (s^1, \dots \widehat{s^j} \dots s^p)$. Expression (79) is seen to be identical to (31) (apart from the trivial choice $[a_k, b_k] = [0,1]$). We thus see from Eq. (31) that

$$\int_{I^p} d\omega = \sum_{q=0,1} \sum_{j=1}^{p} \int_{I(q,j)} \omega_j(x)\, dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^p$$

$$= \int_{\partial I^p} \omega \tag{80}$$

which is the generalisation of (45) for a general p-dimensional cube. When carrying out the surface integral in (80) we have to make sure that we choose the right orientation (67). For the cube this is in fact trivially done directly from (79). However, we can now use (80) to describe less trivial shapes.

Let namely $c : I^k \to M$ be a mapping from a k-dimensional cube to a k-dimensional submanifold of $\mathbb{R}^n$. It then follows from (18) and (80) that for a $(k-1)$-form ω on M we have

$$\int_{c[I^k]} d\omega \overset{(18)}{=} \int_{I^k} c^*(d\omega) = \int_{I^k} d(c^*\omega)$$

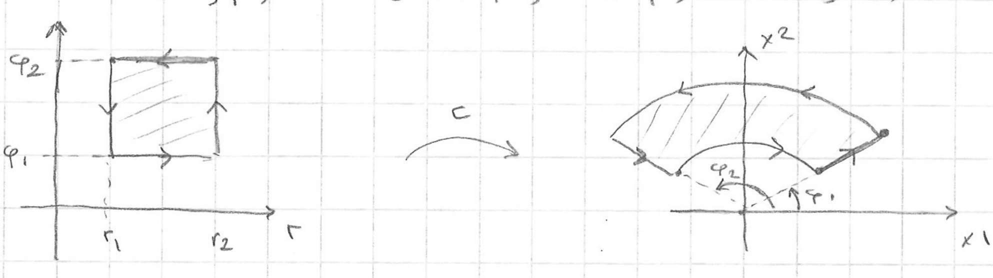
$$\overset{(80)}{=} \int_{\partial I^k} c^*\omega \overset{(18)}{=} \int_{c[\partial I^k]} \omega \qquad (81)$$

where in the second step we used the relation $c^* d\omega = d c^* \omega$ (see eg. Eq. (40) Lecture Notes 8, the proof for a general k-form is identical).

Eq. (81) tells that the integral of the k-form $d\omega$ over the image of the k-cube is the same as integrating the $(k-1)$-form over the image of the boundary of the cube. Let us give some examples.

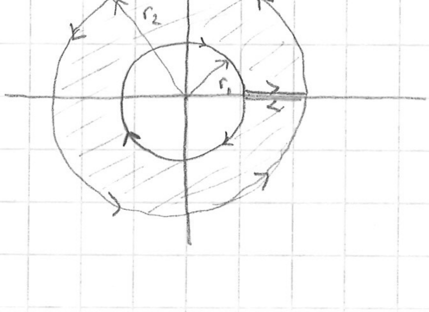
1) Consider the mapping $c : I^2 \to \mathbb{R}$ given by

$$c(r, \varphi) = (r\cos\varphi, r\sin\varphi) = (x^1, x^2)$$



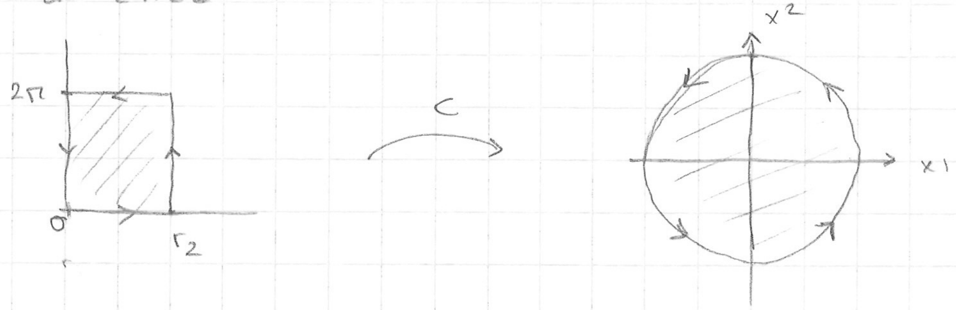
(82)

For $\varphi_1 = 0$, $\varphi_2 = 2\pi$ this becomes



(83)

If we further take $r_1 = 0$ we obtain the interior of a circle

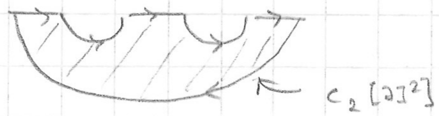


(84)

In general we can combine integrals over images of 2-cubes (squares). If ω is a one-form and we integrate ω over the curves $c_1[\partial I^2]$ and $c_2[\partial I^2]$ given by

$$\int_{c_1[\partial I^2]} \omega$$

and



$$\int_{c_2[\partial I^2]} \omega$$

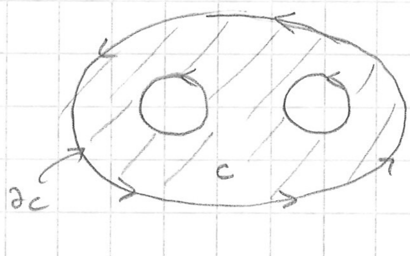
then

$$\int_{\partial c} \omega = \int_c d\omega \tag{85}$$

where $c = c_1[I^2] + c^2[I^2]$ (symbolic addition)

$\partial c = c_1[\partial I^2] + c^2[\partial I^2]$

i.e.



(86)

If $\omega = a(x^1, x^2)\,dx^1 + b(x^1 x^2)\,dx^2$ (87)

Then $d\omega = \left(\frac{\partial b}{\partial x^1} - \frac{\partial a}{\partial x^2}\right) dx^1 \wedge dx^2$ (88)

and (85) tells us that

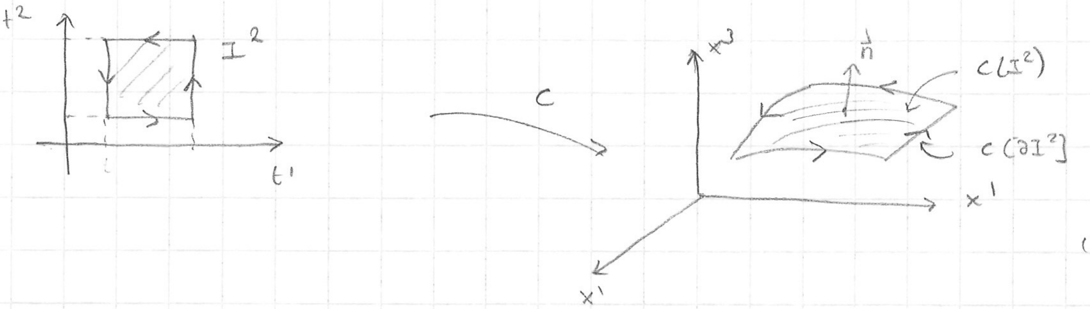
$$\int_{\partial c} (a\,dx^1 + b\,dx^2) = \int_c \left(\frac{\partial b}{\partial x^1} - \frac{\partial a}{\partial x^2}\right) dx^1 dx^2 \tag{89}$$

This is the famous Green's theorem applying to any region c in the plane and its boundary ∂c.

2] This example amounts to the classic Stokes' theorem.

Consider a mapping $c : I^2 \to \mathbb{R}^3$,

$$(x^1, x^2, x^3) = (c^1(t^1,t^2),\ c^2(t^1,t^2),\ c^3(t^1,t^2))$$



(90)

and take $$\omega = \omega_1 dx^1 + \omega_2 dx^2 + \omega_3 dx^3 \qquad (91)$$

Then $$d\omega = \left(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2}\right) dx^1 \wedge dx^2 + \left(\frac{\partial \omega_1}{\partial x^3} - \frac{\partial \omega_3}{\partial x^1}\right) dx^3 \wedge dx^1$$
$$+ \left(\frac{\partial \omega_3}{\partial x^2} - \frac{\partial \omega_2}{\partial x^3}\right) dx^2 \wedge dx^3 \qquad (92)$$

Eq. (85) then tells us that

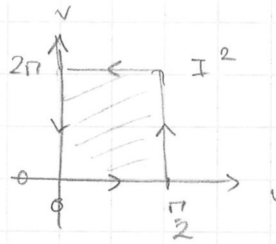
$$\int_{c(I^2)} \left(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2}\right) dx^1 dx^2 + \left(\frac{\partial \omega_1}{\partial x^3} - \frac{\partial \omega_3}{\partial x^1}\right) dx^3 dx^1 + \left(\frac{\partial \omega_3}{\partial x^2} - \frac{\partial \omega_2}{\partial x^3}\right) dx^2 dx^3$$
$$= \int_{c[\partial I^2]} \omega_1 dx^1 + \omega_2 dx^2 + \omega_3 dx^3 \qquad (91)$$

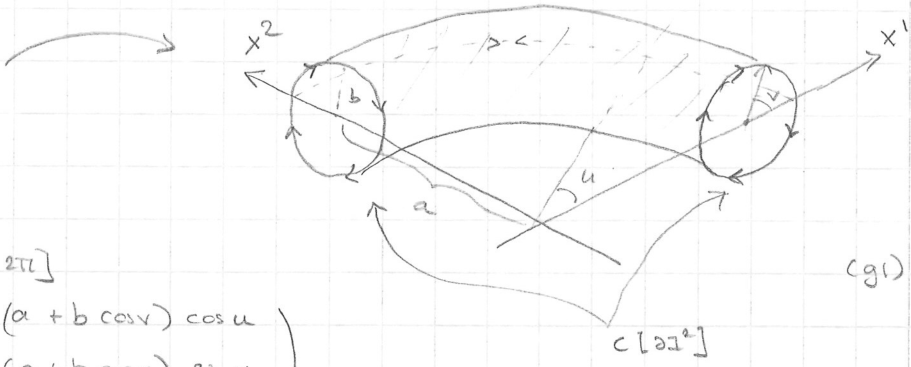
Using (14) and (25) this can be rewritten as

$$\int_{c(I^2)} dA \ (\nabla \times \vec{\omega}) \cdot \vec{n} = \int_{c[\partial I^2]} \langle \vec{\omega}, \frac{d\vec{c}}{dt} \rangle \qquad (90)$$

where $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$, $\vec{n}$ the normal vector to the surface and $\frac{d\vec{c}}{dt}$ the tangent vector to the curve $c(t)$ which parametrizes the edge $c[\partial I^2]$. Eq. (90) is the classic Stokes' theorem.

A nice specific example of a map $c: I^2 \to \mathbb{R}^3$ is given by the torus in (1) of Lecture Notes 5.



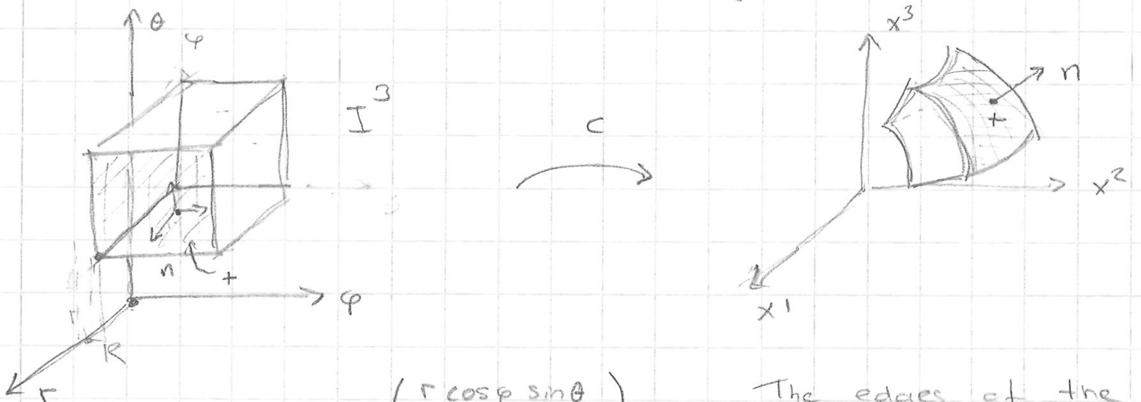


$u \in [0, \frac{\pi}{2}]$, $v \in [0, 2\pi]$ (g1)

$$c(u,v) = \begin{pmatrix} (a + b\cos v)\cos u \\ (a + b\cos v)\sin u \\ b \sin v \end{pmatrix}$$

Stokes' theorem (g0) applied to this example then tells us that the flux of $\nabla \times \vec{u}$ through the torus is equal to the line integral $\int_{\partial c} \omega$ on the edges of the tube which are 'circled' in opposite directions.

3] Let us now look at images of 3-cubes I^3.

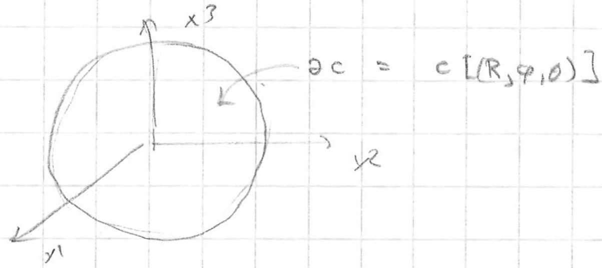


(g2)

$$c(r, \theta, \varphi) = \begin{pmatrix} r\cos\varphi \sin\theta \\ r\sin\varphi \sin\theta \\ r\cos\theta \end{pmatrix}$$

The edges of the cube have orientation $(-1)^{j+\alpha}$.

In the case $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$, $r \in [0, R]$ the image is a 'sphere'



(g3)

The surface of the cube is the image of the (R, θ, φ)-plane. Opposing planes of I^3 have opposite orientation.

We can take the 2-form

$$\omega = \omega_1\, dx^2 \wedge dx^3 + \omega_2\, dx^3 \wedge dx^1 + \omega_3\, dx^2 \wedge dx^3 \tag{94}$$

Then

$$d\omega = \left[\frac{\partial \omega_1}{\partial x^1} + \frac{\partial \omega_2}{\partial x^2} + \frac{\partial \omega_3}{\partial x^3} \right] dx^1 \wedge dx^2 \wedge dx^3 \tag{95}$$

Eq. (85) then tells us that

$$\int_{c[I^3]} \left[\frac{\partial \omega_1}{\partial x^1} + \frac{\partial \omega_2}{\partial x^2} + \frac{\partial \omega_3}{\partial x^3} \right] dx^1 dx^2 dx^3$$

$$= \int_{c[\partial I^3]} \left(\omega_1\, dx^2 dx^3 + \omega_2\, dx^3 dx^1 + \omega_3\, dx^2 dx^3 \right) \tag{96}$$

or equivalently using (25)

$$\int_{c[I^3]} (\vec{\nabla} \cdot \vec{\omega})\, dV = \int_{c[\partial I^3]} (\vec{\omega} \cdot \vec{n})\, dA \tag{97}$$

$\vec{\omega} = (\omega_1, \omega_2, \omega_3)$, where n is the normal vector to the surface.
Eq. (97) is nothing but the famous Gauss' theorem.

We thus see that the classic integral theorems of Green, Stokes and Gauss are all special cases of Eq. (85). Here we always integrated images of k-cubes. It is not difficult to show that any compact smooth manifold can be written as the sum of a finite number of images of k-cubes.
(See M. Spivak, Introduction to Differential Geometry Vol I).
This means that (85) is true for a general manifold M. We can thus write (85) as

$$\int_M d\omega = \int_{\partial M} \omega \tag{98}$$

where ∂M is the edge of the manifold.
This is known as the generalized Stokes theorem.

Let us now apply (98) to some interesting cases. Let ω be a 1-form in n dimensions

$$\omega = \sum_{j=1}^{n} \omega_j(x)\, dx^j \qquad (99)$$

Then according to Eqs. (150) and (151) of Lecture Notes 4 we have

$$\star\omega = \frac{1}{(n-1)!} \sum_{i_2 \ldots i_n}^{n} (\star\omega)_{i_2 \ldots i_n}\, dx^{i_2} \wedge \ldots \wedge dx^{i_n} \qquad (100)$$

where

$$(\star\omega)_{i_2 \ldots i_n} = \sum_{j_1=1}^{n} \omega^{j_1} \epsilon_{j_1 i_2 \ldots i_n} \sqrt{|g|} \qquad (101)$$

(we use general coordinates), or, if we insert (101) into (100)

$$\star\omega = \frac{1}{(n-1)!} \sum_{j_1 i_2 \ldots i_n}^{n} \omega^{j_1} \epsilon_{j_1 i_2 \ldots i_n} \sqrt{|g|}\, dx^{i_2} \wedge \ldots \wedge dx^{i_n} \qquad (102)$$

This presents a $(n-1)$-form in n-dimensions. The $(n-1)$-form (102) is closely related to the volume form Ω

$$\Omega = \sqrt{|g|}\, dx^1 \wedge \ldots \wedge dx^n \qquad (103)$$

If we define the vector

$$\omega^{\#} = \sum_{j=1}^{n} \omega^j \frac{\partial}{\partial x^j} \qquad (104)$$

with $\omega^j = \sum_k^n g^{jk} \omega_k$, and take $(n-1)$ other vectors $v_2, \ldots, v_n$ then

$$\Omega(\omega^{\#}, v_2, \ldots, v_n) = \sqrt{|g|} \sum_{i_1 \ldots i_n}^{n} \epsilon_{i_1 \ldots i_n}\, dx^{i_1} \otimes \ldots \otimes dx^{i_n} (\omega^{\#}, v_2, \ldots, v_n)$$

$$= \sqrt{|g|} \sum_{i_1 \ldots i_n}^{n} \omega^{i_1} \epsilon_{i_1 \ldots i_n}\, dx^{i_2} \otimes \ldots \otimes dx^{i_n} (v_2, \ldots, v_n)$$

$$= \frac{\sqrt{|g|}}{(n-1)!} \sum_{i_1 \ldots i_n}^{n} \omega^{i_1} \epsilon_{i_1 \ldots i_n}\, dx^{i_2} \wedge \ldots \wedge dx^{i_n} (v_2, \ldots, v_n)$$

$$= (\star\omega)(v_2, \ldots, v_n) \qquad (105)$$

This is often denoted as

$$\star\omega = i_{\omega^{\#}} \Omega \qquad (106)$$

Where $i_{\omega^{\#}}$ denotes that $\omega^{\#}$ is inserted in the first argument of Ω. We now apply Stokes' theorem (98) to the $(n-1)$-form $\star\omega$.

$$\int_M d(\star\omega) = \int_{\partial M} \star\omega \qquad (107)$$

where M is a n-dimensional manifold with $(n-1)$-dimensional boundary ∂M. Let us calculate $d\star\omega$ more explicitly. From (102) we have

$$d\star\omega = \frac{1}{(n-1)!}\sum_{i_1\ldots i_n}^{n} \frac{\partial(\omega^{i_1}\sqrt{|g|})}{\partial x^{i_1}} \epsilon_{i_1\ldots i_n}\, dx^{i_1}\wedge\ldots\wedge dx^{i_n}$$

$$= \sum_{i_1=1}^{n} \frac{\partial(\omega^{i_1}\sqrt{|g|})}{\partial x^{i_1}}\, dx^1\wedge\ldots\wedge dx^n$$

$$= \frac{1}{\sqrt{|g|}}\sum_{i_1=1}^{n} \frac{\partial(\omega^{i_1}\sqrt{|g|})}{\partial x^{i_1}}\, \Omega$$

(150) Lec. Notes. 8)

$$= \sum_{i_1=1}^{n} \omega^{i_1}{}_{;i_1}\, \Omega = \operatorname{div}\omega\, \Omega \qquad (108)$$

so $d\star\omega$ is the divergence of ω times the volume form Ω. Therefore (107) becomes

$$\int_M \operatorname{div}\omega\, \Omega = \int_{\partial M} \star\omega \qquad (109)$$

Let us now work out the right hand side of (109) in more detail. If we parametrize the $(n-1)$-dimensional surface ∂M of M by $\varphi(t^1,\ldots,t^{n-1}) = (\varphi^1(t^1,\ldots,t^{n-1}),\ldots,\varphi^n(t^1,\ldots,t^{n-1}))$ then the pull back $\varphi^*(\star\omega)$ of $\star\omega$ under φ will be of the form

$$\varphi^*(\star\omega) = f(t^1,\ldots,t^{n-1})\, dt^1\wedge\ldots\wedge dt^{n-1} \qquad (110)$$

The function f is then determined from

$$f = \varphi^*(\star\omega)\left(\frac{\partial}{\partial t^1},\ldots,\frac{\partial}{\partial t^{n-1}}\right)$$

$$= \star\omega\left(\varphi_*\frac{\partial}{\partial t^1},\ldots,\varphi_*\frac{\partial}{\partial t^{n-1}}\right) \qquad \text{(using (105))}$$

$$= \Omega\left(\omega^\#(\varphi(t)),\varphi_*\frac{\partial}{\partial t^1},\ldots,\varphi_*\frac{\partial}{\partial t^{n-1}}\right)$$

$$= \det\left(\omega^\#, \frac{\partial\varphi}{\partial t^1}, \dots, \frac{\partial\varphi}{\partial t^{n-1}}\right)\sqrt{|g|}$$

$$= \begin{vmatrix} \omega^1 & \frac{\partial\varphi^1}{\partial t^1} & \cdots & \frac{\partial\varphi^1}{\partial t^{n-1}} \\ \vdots & \vdots & & \vdots \\ \omega^n & \frac{\partial\varphi^n}{\partial t^1} & \cdots & \frac{\partial\varphi^n}{\partial t^{n-1}} \end{vmatrix} \sqrt{|g|} \qquad (111)$$

where we used that the push-forward $\varphi_*\left(\frac{\partial}{\partial t^k}\right) = \sum_{l=1}^{n} \frac{\partial\varphi^l}{\partial t^k}\frac{\partial}{\partial x^k}$
We thus have that

$$\int_{\partial M} \star\omega = \int_{\varphi^{-1}(\partial M)} \begin{vmatrix} \omega^1 & \frac{\partial\varphi^1}{\partial t^1} & \cdots & \frac{\partial\varphi^1}{\partial t^{n-1}} \\ \vdots & \vdots & & \vdots \\ \omega^n & \frac{\partial\varphi^n}{\partial t^1} & \cdots & \frac{\partial\varphi^n}{\partial t^{n-1}} \end{vmatrix} \sqrt{|g|}\, dt^1 \dots dt^{n-1} \qquad (112)$$

This is the n-dimensional generalization of Eq. (25).
It describes the flux of the n-dimensional vector $\omega^\#$ through the (n-1)-dimensional surface ∂M.

Let us now, to be more specific, consider the flux of a 4-dimensional vector through a 3-dimensional surface in Minkowski space. We considered such a case in Lecture Notes 13. In that case the vector was one column of the energy-momentum tensor, which would need the consideration of vector valued one-forms to define (see S. Parrott, "Relativistic Electrodynamics and Differential Geometry"). But we could equally do this for current conservation.

Let n then be a vector orthogonal to the tangent vectors $\frac{\partial\varphi}{\partial t^1}, \frac{\partial\varphi}{\partial t^2}, \frac{\partial\varphi}{\partial t^3}$ on the 3-dimensional surface ∂M, i.e.

$$\left\langle n, \frac{\partial\varphi}{\partial t^j}\right\rangle = g\left(n, \frac{\partial\varphi}{\partial t^j}\right) = 0 \qquad j = 1,2,3 \qquad (113)$$

w.r.t. the Minkowski metric. If n is not light like i.e. $\langle n,n\rangle \neq 0$ then it can be normalized

$$\langle n,n\rangle = \pm 1 \qquad (114)$$

with + if n is space-like and − if n is time-like

The vector $\omega^\#$ can then be expanded as

$$\omega^\# = \frac{\langle \omega^\#, n\rangle}{\langle n, n\rangle} n + \sum_{j=1}^{3} g_j \frac{\partial \varphi}{\partial t^j} \tag{115}$$

Inserting this into (112) gives

$$\int_{\partial M} \star\omega = \int_{\varphi^{-1}(\partial M)} \frac{\langle \omega^\#, n\rangle}{\langle n, n\rangle} \begin{vmatrix} n^1 & \frac{\partial\varphi^1}{\partial t^1} & \frac{\partial\varphi^1}{\partial t^2} & \frac{\partial\varphi^1}{\partial t^3} \\ \vdots & \vdots & \vdots & \vdots \\ n^4 & \frac{\partial\varphi^4}{\partial t^1} & \frac{\partial\varphi^4}{\partial t^2} & \frac{\partial\varphi^4}{\partial t^3} \end{vmatrix} \sqrt{|g|}\, dt^1 dt^2 dt^3$$

$$= \int_{\varphi^{-1}(\partial M)} \frac{\langle \omega^\#, n\rangle}{\langle n, n\rangle} d\sigma \tag{116}$$

where

$$d\sigma = \begin{vmatrix} n^1 & \frac{\partial\varphi^1}{\partial t^1} & \frac{\partial\varphi^1}{\partial t^2} & \frac{\partial\varphi^1}{\partial t^3} \\ n^2 & \frac{\partial\varphi^2}{\partial t^1} & \frac{\partial\varphi^2}{\partial t^2} & \frac{\partial\varphi^2}{\partial t^3} \\ n^3 & \frac{\partial\varphi^3}{\partial t^1} & \frac{\partial\varphi^3}{\partial t^2} & \frac{\partial\varphi^3}{\partial t^3} \\ n^4 & \frac{\partial\varphi^4}{\partial t^1} & \frac{\partial\varphi^4}{\partial t^2} & \frac{\partial\varphi^4}{\partial t^3} \end{vmatrix} \sqrt{|g|}\, dt^1\wedge dt^2\wedge dt^3 \tag{117}$$

describes the "surface" volume element. With (116) and (109) we can thus write the equivalent of Gauss' theorem in Minkowski space as

$$\int_M \operatorname{div}\omega\, \Omega = \int_{\varphi^{-1}(\partial M)} \frac{\langle \omega^\#, n\rangle}{\langle n, n\rangle} d\sigma \tag{118}$$

The right hand side of (118) is also often written as

$$\int_{\varphi^{-1}(\partial M)} \frac{\langle \omega^\#, n\rangle}{\langle n, n\rangle} d\sigma = \sum_{i=1}^{4} \int \omega_i\, d\sigma^i \tag{119}$$

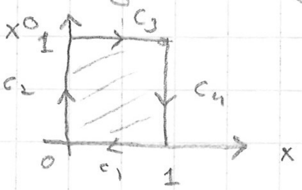
where $d\sigma^i = \epsilon\, n^i\, d\sigma$ (120)

where $\epsilon = \mp 1$ for space / time-like surfaces respectively. In case the surface is light-like Eq. (116) can not be used and we have to use (112) directly instead.
In fact, this is often the simplest thing to do in all cases in practice.

Let us see how (119) works in an example. For simplicity we take a 2-dimensional Minkowski space with metric

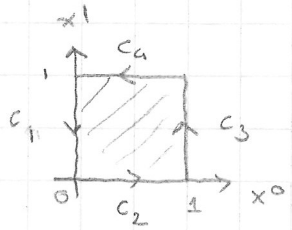
$$g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{121}$$

We integrate along the path



A

equivalent to



B (122)

The orientation in A looks wrong but this is due to the habit in relativity to draw x^0 on the y-axis despite the fact that it occurs as the first coordinate in (x^0, x^1).
We take the 1-form

$$\omega = \omega_0 dx^0 + \omega_1 dx^1 \tag{113}$$

Then since $|g| = 1$ we have that $\omega^\# = (\omega^0, \omega^1)$ and

$$\mathrm{div}\,\omega\,\Omega = \left(\frac{\partial \omega^0}{\partial x^0} + \frac{\partial \omega^1}{\partial x^1} \right) dx^0 \wedge dx^1 \tag{114}$$

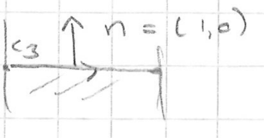
Let $M = [0,1] \times [0,1]$ be the interior of the square in (122) A. Then

$$\int_M \mathrm{div}\,\omega\,\Omega = \int_0^1 dx^0 \int_0^1 dx^1 \left[\frac{\partial \omega^0}{\partial x^0} + \frac{\partial \omega^1}{\partial x^1} \right]$$

$$= \int_0^1 dx^1 \left[\omega^0(1, x^1) - \omega^0(0, x^1) \right] + \int_0^1 dx^0 \left[\omega^1(x^0, 1) - \omega^1(x^0, 0) \right] \tag{115}$$

For the integral along the time-like curve c_3 we have the parametrization

$$\varphi(s) = (1, s), \qquad n = (1, 0)$$



Then

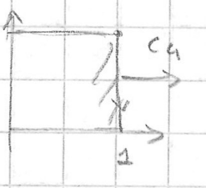
$$d\sigma = \begin{vmatrix} n^0 & \frac{\partial \varphi^0}{\partial s} \\ n^1 & \frac{\partial \varphi^1}{\partial s} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} ds = ds \tag{116}$$

$$\langle \omega, n \rangle = \langle (\omega^0, \omega^1), (1,0) \rangle = -\omega^0 \qquad (117)$$
$$\langle n, n \rangle = -1$$

Then $$\int_{c_3} \frac{\langle \omega, n\rangle}{\langle n,n\rangle} \, d\sigma = \int_0^1 ds \; \omega^0(1,s) \qquad (118)$$

Similarly one finds $$\int_{c_1} \frac{\langle \omega, n\rangle}{\langle n,n\rangle} \, d\sigma = -\int_0^1 ds \; \omega^0(0,s) \qquad (119)$$

The curve c_4 is a space-like curve with normal vector $n = (0, 1)$



$$\varphi(s) = (1-s, 1) \qquad (120)$$

Then $$d\sigma = \begin{vmatrix} n^0 & \frac{\partial \varphi^0}{\partial s} \\ n^1 & \frac{\partial \varphi^1}{\partial s} \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} ds = ds$$

$$\langle \omega, n \rangle = \langle (\omega^0, \omega^1), (0,1) \rangle = \omega^1 \qquad (121)$$
$$\langle n, n \rangle = 1$$

Then $$\int_{c_4} \frac{\langle \omega, n\rangle}{\langle n,n\rangle} \, d\sigma = \int_0^1 ds \; \omega^1(1-s, 1)$$
$$\overset{\tau = 1-s}{=} \int_0^1 d\tau \; \omega^1(\tau, 1) \qquad (122)$$

Similarly we find $$\int_{c_2} \frac{\langle \omega, n\rangle}{\langle n,n\rangle} \, d\sigma = -\int_0^1 ds \; \omega^1(s, 0) \qquad (123)$$

Adding (118), (119), (122), (123) then gives

$$\int_{\varphi^{-1}[\partial M]} \frac{\langle \omega, n\rangle}{\langle n,n\rangle} \, d\sigma$$

$$= \int_0^1 ds \left[\omega^0(1,s) - \omega^0(0,s) \right] + \int_0^1 ds \left[\omega^1(s,1) - \omega^1(s,0) \right] \qquad (124)$$

which is the same expression as in (115).

In electromagnetism obvious application of Stokes' theorem involves the equations $F = dA$ and $\star d \star F = \frac{4\pi}{c} J$. This means that

$$\int_M F = \int_M dA = \int_{\partial M} A \qquad (125)$$

where M is a 2-dimensional manifold in 4 dimensions. Explicitly we have

$$F = -E_1\, dx^0 \wedge dx^1 - E_2\, dx^0 \wedge dx^2 - E_3\, dx^0 \wedge dx^3$$

$$+ B_1\, dx^2 \wedge dx^3 + B_2\, dx^3 \wedge dx^1 + B_3\, dx^1 \wedge dx^2 \qquad (126)$$

$$A = A_0\, dx^0 + A_1\, dx^1 + A_2\, dx^2 + A_3\, dx^3 \qquad (127)$$

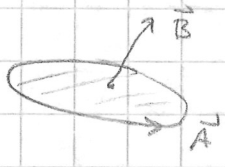
Eq. (125) is often used for the case of an equal-time surface, i.e. $dx^0 = 0$. Then (125) becomes

$$\int_M (B_1\, dx^2 \wedge dx^3 + B_2\, dx^3 \wedge dx^1 + B_3\, dx^1 \wedge dx^2)$$

$$= \int_{\partial M} (A_1 dx^1 + A_2 dx^2 + A_3 dx^3) \qquad (128)$$

more commonly written as

$$\int \vec{B} \cdot d\vec{S} = \int \vec{A} \cdot d\vec{\ell}$$



Further from $\star d \star F = \frac{4\pi}{c} J$ it follows that

$$\star \star d \star F = \frac{4\pi}{c} \star J \qquad (129)$$

Now $d \star F$ is a 3-form. It then follows from Eq. (167) of Lecture Notes 4 that

$$\star \star d \star F = (-1)^{3 \cdot (4-1)} \operatorname{sgn}(g)\, d \star F = d \star F \qquad (130)$$

and thus $$d \star F = \frac{4\pi}{c} \star J \qquad (131)$$

With Stokes' theorem it then follows that

$$\frac{4\pi}{c}\int_M \star J = \int_M d\star F = \int_{\partial M} \star F \qquad (132)$$

We have (see (116), Lecture Notes 8)

$$\star F = B_1\, dx^0 \wedge dx^1 + B_2\, dx^0 \wedge dx^2 + B_3\, dx^0 \wedge dx^3$$
$$+ E_1\, dx^2 \wedge dx^3 + E_2\, dx^3 \wedge dx^1 + E_3\, dx^1 \wedge dx^2 \qquad (133)$$

Further the current 4-vector is $j^i = (c\rho, \vec{j}) = (j^0, j^1, j^2, j^3)$. Then

$$J = j_0\, dx^0 + j_1\, dx^1 + j_2\, dx^2 + j_3\, dx^3 \qquad (134)$$

We further have (see (143) Lect. Notes 4)

$$\begin{aligned}
\star dx^0 &= \epsilon_{0123}\, g^{00} \sqrt{|g|}\, dx^1 \wedge dx^2 \wedge dx^3 \\
&= -dx^1 \wedge dx^2 \wedge dx^3 \\
\star dx^1 &= \epsilon_{1023}\, g^{11} \sqrt{|g|}\, dx^0 \wedge dx^2 \wedge dx^3 \\
&= -dx^0 \wedge dx^2 \wedge dx^3 \\
\star dx^2 &= \epsilon_{2013}\, g^{22} \sqrt{|g|}\, dx^0 \wedge dx^1 \wedge dx^3 \\
&= -\sqrt{|g|}\, dx^0 \wedge dx^3 \wedge dx^1 \\
\star dx^3 &= \epsilon_{3012}\, g^{33} \sqrt{|g|}\, dx^0 \wedge dx^1 \wedge dx^2 \\
&= -\sqrt{|g|}\, dx^0 \wedge dx^1 \wedge dx^2
\end{aligned} \qquad (135)$$

and therefore

$$\star J = -j_0 \sqrt{|g|}\, dx^1 \wedge dx^2 \wedge dx^3 - j_1 \sqrt{|g|}\, dx^0 \wedge dx^2 \wedge dx^3$$
$$- j_2 \sqrt{|g|}\, dx^0 \wedge dx^3 \wedge dx^1 - j_3 \sqrt{|g|}\, dx^0 \wedge dx^1 \wedge dx^2 \qquad (136)$$

In the case that we take a 3-dimensional volume in Minkowski space with $dx^0 = 0$. Then using $j_0 = -c\rho$ Eq. (132) becomes

$$4\pi \int_M \rho \sqrt{|g|}\, dx^1 \wedge dx^2 \wedge dx^3$$
$$= \int_{\partial M} E_1\, dx^2 \wedge dx^3 + E_2\, dx^3 \wedge dx^1 + E_3\, dx^1 \wedge dx^2 \qquad (137)$$

which is more commonly written as

$$4\pi \int_M \rho \, dV = \int_{\partial M} \vec{E} \cdot d\vec{s} \tag{138}$$

i.e. standard Gauss' law.

We now come to a final useful application of the integration of differential forms. Remember from Eq. (119) of Lecture Notes 4, that if ω, μ are p-forms then

$$\omega \wedge \star \mu = \langle \omega, \mu \rangle \Omega \tag{139}$$

Now since Ω is a volume form it is natural to integrate over it. We can thus define a new inner product

$$(\omega, \mu) \equiv \int_M \langle \omega, \mu \rangle \Omega = \int_M \omega \wedge \star \mu \tag{140}$$

where we integrate over a manifold M.
Let ω now a (p-1)-form and μ a p-form.
Then $d\omega$ is a p-form and according to (140) we have

$$(d\omega, \mu) = \int_M d\omega \wedge \star \mu \tag{141}$$

Let us now calculate

$$d(\omega \wedge \star \mu) = d\omega \wedge \star \mu + (-1)^{p-1} \omega \wedge d \star \mu \tag{142}$$

where we used Eq. (72) of Lecture Notes 8.
Now $d \star \mu$ is a (n-p+1)-form and therefore according to (167) of Lecture Notes 4 we have

$$\begin{aligned} \star \star d \star \mu &= (-1)^{(n-p+1)(p-1)} \operatorname{sign}(g) \, d \star \mu \\ &= (-1)^{np+n+p+1} \operatorname{sign}(g) \, d \star \mu \end{aligned} \tag{143}$$

Using this expression we can write (142) as

$$d(\omega \wedge \star \mu) = d\omega \wedge \star \mu - (-1)^{np+n+1} \operatorname{sign}(g) \, \omega \wedge \star \star d \star \mu \tag{144}$$

If we now remember that in (68) of Lecture Notes 9 we defined the operator

$$d^{+} = \operatorname{sign}(g)\,(-1)^{np+n+1} \star d \star \tag{145}$$

then we see that (144) can be written as

$$d(\omega \wedge \star\mu) = d\omega \wedge \star\mu - \omega \wedge \star d^{+}\mu \tag{146}$$

Integrating (146) over the manifold M gives

$$\begin{aligned}\int_M d(\omega \wedge \star\mu) &= \int_M d\omega \wedge \star\mu - \int_M \omega \wedge \star d^{+}\mu \\ &= (d\omega, \mu) - (\omega, d^{+}\mu)\end{aligned} \tag{147}$$

Using Stokes' theorem for the term on the lefthand side of (147) we finally find that

$$(d\omega, \mu) = (\omega, d^{+}\mu) + \int_{\partial M} \omega \wedge \star\mu \tag{148}$$

If ω or $\star\mu$ vanishes on the boundary ∂M of M, or if M has no boundary (such as the surface of a sphere) then the last term in (148) is zero and (148) yields

$$(d\omega, \mu) = (\omega, d^{+}\mu) \tag{149}$$

Therefore the operator d^{+} is the adjoint operator of d with respect to the inner product (140). More explicitly (149) is given by

$$\int_M d\omega \wedge \star\mu = \int_M \omega \wedge \star d^{+}\mu \tag{150}$$

Equations (148) and (150) will be useful in manipulating action functionals.