Maxwell action and gauge curvature

All the physics that we discussed in this course can be summarized in the Maxwell equations

$$dF = 0 \tag{1}$$

$$\star d \star F = \frac{4\pi}{c} J \tag{2}$$

together with the Lorentz force

$$\left(\frac{Dp}{d\tau}\right)_\mu = \frac{q}{c}\sum_{\nu=0}^{3} F_{\mu\nu} v^\nu \tag{3}$$

where v and p are the 4-velocity and 4-momentum of a particle and τ the proper time.

The natural that arises is whether there is an action principle from which Eqs. (1)-(3) can be derived. This is, of course, the case otherwise we would not be mentioning it.
Let us start by deriving (1) and (2). It is clear that the action should depend on F. Moreover since Eqs. (1) and (2) are linear in F we expect the action to be quadratic in F. Further we expect that it will involve integration over the invariant volume $\Omega = \sqrt{|g|}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$. It is therefore not difficult to guess that the action involves integration over

$$F \wedge \star F = \langle F, F \rangle \Omega \tag{4}$$

Further the action must be linear in J. However, since J is a one-form, it must be combined with a 3-form. Alternatively we can take the action to be linear in $\star J$ which is a 3-form. We then must seek a 1-form to combine it with.
Since $F = dA$, the 1-form A seems a natural choice. Then

$$A \wedge {\star} J = \langle A, J \rangle \, \Omega \qquad (5)$$

Then the total action must be of the form

$$S = \int_M \alpha \, F \wedge {\star} F + \beta \, A \wedge {\star} J \qquad (6)$$

where the relation between α and β will be determined by (2). Since in (2) the current J is given, A must be the variational parameter. We thus write

$$S[A] = \int_M \alpha \, dA \wedge {\star} dA + \beta \, A \wedge {\star} J \qquad (7)$$

Here M is simply all of 4-dimensional space-time. We now make variations δA in (7) which vanish at ∂M at infinity. Then

$$S[A + \delta A] = S[A]$$

$$+ \int_M \alpha \left(d\delta A \wedge {\star} dA + dA \wedge {\star} d\delta A \right.$$

$$\left. + \beta \, \delta A \wedge {\star} J \right) \qquad (8)$$

Using

$$\int \omega \wedge {\star} \mu = \int \langle \omega, \mu \rangle \, \Omega$$

$$= \int \langle \mu, \omega \rangle \, \Omega = \int \mu \wedge {\star} \omega \qquad (9)$$

for the second term under the integral we have

$$S[A + \delta A] - S[A]$$

$$+ \int_M 2\alpha \, d\delta A \wedge {\star} dA + \beta \, \delta A \wedge {\star} J$$

$$= \int_M \delta A \wedge \left(2\alpha \, {\star} d^{+} dA + \beta \, {\star} J \right) \qquad (10)$$

where we used Eq. (150) of Lecture Notes 13

$$\int_M d\omega \wedge \star\mu = \int_M \omega \wedge \star d^+\mu \tag{11}$$

valid if ω and/or $\star\mu$ vanish at ∂M.
From (10) we see that if $\delta S = 0$ for all possible variations δA then we must have

$$2\alpha \star d^+ dA + \beta \star J = 0 \tag{12}$$

This gives, since $dA = F$

$$d^+ F = -\frac{\beta}{2\alpha} J \tag{13}$$

In 4-dimensional space-time $d^+ = \star d \star$ (see (84) Lecture Notes 9) and hence (13) becomes

$$\star d \star F = -\frac{\beta}{2\alpha} J \tag{14}$$

where $F = dA$ and hence $dF = 0$. This yields exactly Eqs. (1) and (2) provided $\beta/2\alpha = -\frac{4\pi}{c}$ or $\beta = -\frac{8\pi\alpha}{c}$. We can therefore write the action as

$$S[A] = \alpha \int_M \left(F \wedge \star F - \frac{8\pi}{c} A \wedge \star J\right) \tag{15}$$

Using Eqs. (4) and (5) this can also be written as

$$S[A] = \alpha \int_M \left(\langle F, F\rangle - \frac{8\pi}{c} \langle A, J\rangle\right) \Omega \tag{16}$$

We further have (see (32), Lecture Notes 4)

$$\langle F, F \rangle = \sum_{\mu<\nu} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} \sum_{\mu\nu} F_{\mu\nu} F^{\mu\nu} \tag{17}$$

$$\langle A, J \rangle = \sum_{\mu} A_\mu J^\mu = \sum_\mu J_\mu A^\mu \tag{18}$$

and thus

$$S[A] = \alpha \int_M \left(\sum_{\mu\nu} \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - \frac{8\pi}{c} \sum_\mu J_\mu A^\mu \right) \sqrt{|g|}\, dx^0 dx^1 dx^2 dx^3 \tag{19}$$

The constant α can in principle be chosen arbitrarily but a standard choice is $\alpha = -1/2$ (then a Legendre transformation yields the Hamiltonian of the electromagnetic field). With this choice

$$S[A] = \int_M \left(\sum_{\mu\nu} -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{4\pi}{c} \sum_\mu J_\mu A^\mu \right) \sqrt{|g|}\, dx^0 dx^1 dx^2 dx^3 \tag{20}$$

whereas the compact form (15) becomes

$$S[A] = \int_M \left(-\frac{1}{2} F \wedge \star F + \frac{4\pi}{c} A \wedge \star J \right) \tag{21}$$

We further note that (21) is gauge invariant. If we make a gauge transformation

$$A' = A + d\Lambda \tag{22}$$

then (with $\Lambda = 0$ on ∂M)

$$\begin{aligned} S[A'] &= S[A + d\Lambda] \\ &= \int_M \left(-\frac{1}{2} F \wedge \star F + \frac{4\pi}{c} (A + d\Lambda) \wedge \star J \right) \\ &= S[A] + \frac{4\pi}{c} \int_M d\Lambda \wedge \star J \\ &= S[A] + \frac{4\pi}{c} \int_M \Lambda \wedge \star d^\dagger J \\ &= S[A] \end{aligned} \tag{23}$$

since (see Eq. (69) Lecture Notes 9)

$$d^\dagger J = -\sum_{\mu=0}^{3} j^\mu{}_{;\mu} = 0 \tag{24}$$

as a consequence of charge conservation. There is therefore a close relation between gauge invariance and charge conservation. We have succeeded in deriving Maxwell's equations (1) and (2) from an action principle. It remains to derive the Lorentz force equation (3) as well from an action.

Let us first write out the $A \wedge * \mathcal{J}$ interaction term for a single particle. Let the particle move along a world line $z^\mu(\tau)$ parametrized by the proper time τ. Then according to Eq. (31) of Lecture Notes 10 the 4-current is given by

$$j^\mu(x) = cq \int d\tau \, \frac{dz^\mu}{d\tau} \, \delta^{(4)}(x - z(\tau)) \qquad (25)$$

This equation is true in standard Minkowskian coordinates. In general coordinates we have

$$j^\mu(x) = \frac{cq}{\sqrt{|g|}} \int d\tau \, \frac{dz^\mu}{d\tau} \, \delta^{(4)}(x - z(\tau)) \qquad (26)$$

(for the same reason as discussed below Eq. (158) of Lecture Notes 10). With Eq. (26) the interaction term in (20) is given by

$$\frac{4\pi}{c} \sum_\mu \int \mathcal{J}_\mu A^\mu \sqrt{|g|} \, dx^0 dx^1 dx^2 dx^3$$

$$= \frac{4\pi}{c} \sum_\mu \int j^\mu A_\mu \sqrt{|g|} \, dx^0 dx^1 dx^2 dx^3$$

$$= 4\pi q \sum_\mu \int d\tau \int d^4x \, \frac{dz^\mu}{d\tau} A_\mu(x) \, \delta^{(4)}(x - z(\tau))$$

$$= 4\pi q \sum_\mu \int d\tau \, \frac{dz^\mu}{d\tau} A_\mu(z(\tau)) \qquad (27)$$

So the interaction between the field and the particle is known. The only thing we need to do is to add the relativistic action of a free particle. This must be a scalar defined on a world. The simplest scalar is just the proper time

$$d\tau = \sqrt{-\sum_{\mu\nu} g_{\mu\nu}(z(s)) \frac{dz^\mu}{ds} \frac{dz^\nu}{ds}} \, ds \qquad (28)$$

where s is an arbitrary parameter parametrizing the world line. We thus take the free particle action to be

$$S_0[z] = \alpha \int_{s_0}^{s_1} ds \sqrt{-\sum_{\mu\nu} g_{\mu\nu}(z(s)) \frac{dz^\mu}{ds} \frac{dz^\nu}{ds}} \qquad (29)$$

It is important to note that it is advantageous to parametrize $z^\mu(s)$ with a general path parameter rather than the proper τ since the proper time is defined by the world line as soon as we make variations $z^\mu \to z^\mu + \delta z^\mu$ the formula for the proper time will change to a new one for the deformed path. We therefore also rewrite (27) in terms of a general path parameter s using

$$\int d\tau \, \frac{dz^\mu}{d\tau} A_\mu(z(\tau)) = \int ds \, \frac{d\tau}{ds} \frac{dz^\mu}{d\tau} A_\mu(z(\tau))$$

$$= \int ds \, \frac{dz^\mu}{ds} A(z(\tau(s))) \tag{30}$$

Thus, the action of a particle moving in an electromagnetic field with vector potential A_μ is

$$S_m[z] = \int_{s_0}^{s_1} ds \left[\alpha \sqrt{-g(\dot{z},\dot{z})} + 4\pi q \sum_\mu \dot{z}^\mu A_\mu(z) \right] \tag{31}$$

where $\dot{z}^\mu = \frac{dz^\mu}{ds}$.

The constant α will be determined by the requirement that $\delta S_m = 0$ yields the correct equation (3) for the Lorentz force.
The variational equations are given by

$$0 = \frac{\partial \mathcal{L}}{\partial z^\rho} - \frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}^\rho}\right) \tag{32}$$

with

$$\mathcal{L}(z,\dot{z}) = \alpha \sqrt{-g(\dot{z},\dot{z})} + 4\pi q \sum_\mu \dot{z}^\mu A_\mu \tag{33}$$

We have

$$\frac{\partial \mathcal{L}}{\partial z^\rho} = -\frac{1}{2}\alpha \frac{1}{\sqrt{-g}} \frac{\partial g}{\partial z^\rho} + 4\pi q \sum_\mu \dot{z}^\mu \frac{\partial A_\mu}{\partial z^\rho} \tag{34}$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{z}^\rho} = -\frac{1}{2}\alpha \frac{1}{\sqrt{-g}} \frac{\partial g}{\partial \dot{z}^\rho} + 4\pi q A_\rho \tag{35}$$

$$\Rightarrow \frac{d}{ds}\frac{\partial \mathcal{L}}{\partial \dot{z}^\rho} = \frac{1}{4}\alpha (-g)^{-3/2} \frac{d}{ds}(\sqrt{-g}) \frac{\partial g}{\partial \dot{z}^\rho} - \frac{1}{2}\alpha \frac{1}{\sqrt{-g}} \frac{d}{ds}\frac{\partial g}{\partial \dot{z}^\rho}$$
$$+ 4\pi q \sum_\mu \frac{\partial A_\rho}{\partial z^\mu} \dot{z}^\mu \tag{36}$$

where further

$$\frac{\partial g}{\partial z^\rho} = \frac{\partial}{\partial z^\rho}\sum_{\mu\nu} g_{\mu\nu}\dot z^\mu \dot z^\nu = \sum_{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial z^\rho}\dot z^\mu \dot z^\nu \tag{37}$$

$$\frac{\partial g}{\partial \dot z^\rho} = \frac{\partial}{\partial \dot z^\rho}\sum_{\mu\nu} g_{\mu\nu}\dot z^\mu \dot z^\nu = 2\sum_\nu g_{\rho\nu}\dot z^\nu \tag{38}$$

$$\frac{d}{ds}\frac{\partial g}{\partial \dot z^\rho} = 2\frac{d}{ds}\sum_\nu g_{\rho\nu}\dot z^\nu = 2\sum_\nu g_{\rho\nu}\ddot z^\nu + 2\sum_{\mu\nu}\frac{\partial g_{\rho\nu}}{\partial z^\mu}\dot z^\mu \dot z^\nu \tag{39}$$

Collecting our results we have from (32) that

$$0 = -\frac{\alpha}{2\sqrt{-g}}\sum_{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial z^\rho}\dot z^\mu \dot z^\nu + 4\pi q\sum_\mu\left(\frac{\partial A_\mu}{\partial z^\rho} - \frac{\partial A_\rho}{\partial z^\mu}\right)\dot z^\mu$$

$$+ \frac{\alpha}{2\sqrt{-g}}\left(2\sum_\nu g_{\rho\nu}\ddot z^\nu + \sum_{\mu\nu}\left\{\frac{\partial g_{\rho\nu}}{\partial z^\mu} + \frac{\partial g_{\rho\mu}}{\partial z^\nu}\right\}\dot z^\mu \dot z^\nu\right)$$

$$- \frac{\alpha}{4}(-g)^{-3/2}\frac{\partial g}{\partial \dot z^\rho}\frac{d}{ds}(\sqrt{-g}) \tag{40}$$

Now we have an equation for the stationary point and are allowed to choose a convenient path parameter. From (28) we see that

$$\frac{de}{ds} = \sqrt{-g} \tag{41}$$

and therefore if we choose $s = e$ then $\sqrt{-g} = 1$. In this case Eq. (40) simplifies to

$$0 = \alpha\left[\sum_\nu g_{\rho\nu}\ddot z^\nu + \frac{1}{2}\sum_{\mu\nu}\left\{\frac{\partial g_{\rho\nu}}{\partial z^\mu} + \frac{\partial g_{\rho\mu}}{\partial z^\nu} - \frac{\partial g_{\mu\nu}}{\partial z^\rho}\right\}\dot z^\mu \dot z^\nu\right]$$

$$+ 4\pi q\sum_\mu F_{\rho\mu}\dot z^\mu \tag{42}$$

We recognize in (42) the Christoffel symbols of the first kind and can therefore rewrite (42) as

$$-\alpha\left[\sum_\nu g_{\rho\nu}\ddot z^\nu + \sum_{\mu\nu}[\mu\nu,\rho]\dot z^\mu \dot z^\nu\right]$$

$$= 4\pi q\sum_\mu F_{\rho\mu}\dot z^\mu \tag{43}$$

In other words

$$m\left(\frac{Dv}{dc}\right)_\rho = -\frac{4\pi q m}{\alpha}\sum_\mu F_{\rho\mu}v^\mu \qquad (44)$$

where $v^\mu = \dot{z}^\mu$.

We recover the Lorentz force law when we choose $\alpha = -4\pi mc$. Therefore the required action has the form

$$S_m[z] = \int_{s_0}^{s_1} ds \left[-4\pi mc\sqrt{-g} + 4\pi q \sum_\mu \dot{z}^\mu A_\mu\right] \qquad (45)$$

The total action of field + matter is thus given by

$$S[A,z] = -\frac{1}{4}\sum_{\mu\nu}\int F_{\mu\nu}F^{\mu\nu}\sqrt{|g|}\,dx^0dx^1dx^2dx^3$$
$$+ 4\pi\int_{s_0}^{s_1} ds\left[-mc\sqrt{-g} + q\sum_\mu \dot{z}^\mu A_\mu\right] \qquad (46)$$

The separate variations in z and A then yield all equations (1), (2) and (3). The action (46) is further readily generalized to the case that we have several particles present. We simply has to replace $S_m[z]$ in (45) by

$$S_m^N[\{z^{(k)}\}] = \sum_{k=1}^N S_m[z^{(k)}] \qquad (47)$$

where $z^{(k)}$ is the trajectory of particle k.

We have reached another highlight of this course and derived an action principle from which all our equations of motion can be derived.
What remains is to try to get a better insight into the deeper meaning of the form of the action (46). The derivation of (10) for example, used some flashy techniques of differential geometry. So may look for a deeper geometrical meaning.

Gauge curvature

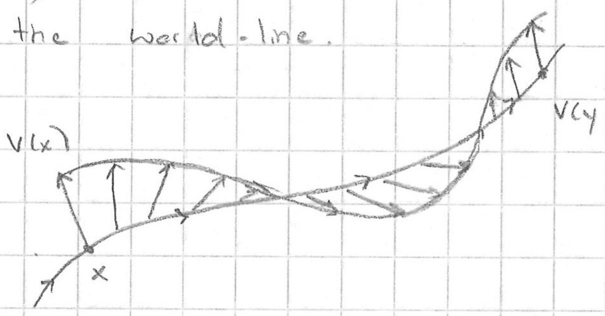
So far we have heavily used the electromagnetic field tensor F but we have not giving a deeper interpretation apart from the fact that its components represent the electric and magnetic fields. This we do here. It will at the same time give a deeper understanding of the gauge invariance property that we have been using.

We now need to let go our interpretation of charged particles as classical particles. We regard them as quantum particles with certain internal properties. The fact that they have internal properties is described by the fact that the particles are described by a space-time dependent vector $v(x)$

$$v(x) = (v^1(x), \ldots, v^M(x)) \tag{48}$$

where M is arbitrary. In particular M is independent of the number of space-time dimensions.

Let now the particle move through space-time along a world-line. Due to the presence of external force fields (electromagnetic fields in the case of electrons) its internal state will change if we move along the world-line.

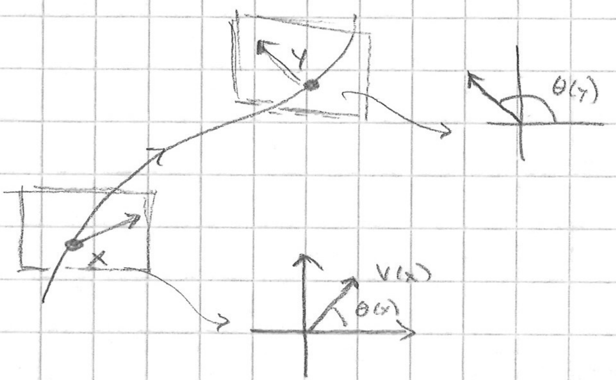


(49)

It is easiest to imagine that vector is rotated as we move along the world-line but we must remember that the rotation is in internal space not in space-time (actually general relativity is an exception to this).

So at every space-time point there is an internal vector space of dimension M attached. Mathematicians call such a thing a fibre bundle

For example, at each space-time point we could attach a 2-dimensional space in which the internal state vector is rotated



(50)

where the rotation angle $\theta(x)$ is determined by the force fields present.

We now give the system a lot of symmetry. We say that there exists a symmetry group G with space-time dependent group elements $g(x) \in G$ and we say that

$$v'(x) = g(x)\, v(x) \tag{51}$$

is a physically equivalent vectorfield for any choice $g(x) \in G$ (but we assume that $g(x)$ is a continuous and differentiable function of x), i.e. indistinguishable in its predictions of observables.
As an example we can take example (50) again of a 2-dimensional vector field with group G given by

$$g(x) = \begin{pmatrix} \cos\theta(x) & \sin\theta(x) \\ \sin\theta(x) & -\cos\theta(x) \end{pmatrix} \tag{52}$$

where $\theta(x)$ an arbitrary function. This is the group of orthogonal 2×2-matrices with determinant 1, also known as $SO(2)$. Other common groups in physics are for instance the unitary $n\times n$-matrices of determinant 1, known as $SU(n)$. Using (52) we this have that

$$\begin{pmatrix} v^1(x) \\ v^2(x) \end{pmatrix}' = \begin{pmatrix} \cos\theta(x) & \sin\theta(x) \\ \sin\theta(x) & -\cos\theta(x) \end{pmatrix} \begin{pmatrix} v^1(x) \\ v^2(x) \end{pmatrix} \tag{53}$$

is a vector field with the same physical predictions as $v(x) = (v^1(x), v^2(x))$

The set of vector fields $v(x)$ is thus split into equivalence classes of vector fields related by symmetry operations $g(x)$, in which each class (also called gauge-class) gives different physical predictions.
For example, 2 vector fields v and v' with a different Euclidean lengt $\|v\|^2 = (v^1)^2 + (v^2)^2$ is some space-time point x can not be related by symmetry (53).

Let us now select an element $v(x)$ of a given gauge class and consider that we move in space-time from point x to point y, as in (50), along a curve c. The vector $v(x)$ is then mapped to a vector $v(y)$ by this operation

$$v(y) = \Gamma_c(y,x)\, v(x) \tag{54}$$

The operator $\Gamma_c(y,x)$ is some kind of evolution operator or propagator that maps a vector on a curve to another vector on the curve. For the transformed field $v'(x) = g(x)\,v(x)$ we have a similar operator $\Gamma'_c(y,x)$ given by

$$v'(y) = \Gamma'_c(y,x)\, v'(x) \tag{55}$$

or equivalently

$$g(y)\, v(y) = \Gamma'_c(y,x)\, g(x)\, v(x)$$

$$\Longrightarrow \quad v(y) = g^{-1}(y)\, \Gamma'_c(y,x)\, g(x)\, v(x) \tag{56}$$

Comparing to (54) we thus have that

$$\Gamma_c(y,x) = g^{-1}(y)\, \Gamma'_c(y,x)\, g(x) \tag{57}$$

or

$$\Gamma'_c(y,x) = g(y)\, \Gamma(y,x)\, g^{-1}(x) \tag{58}$$

It is further clear that

$$\Gamma_c^{-1}(y,x) = \Gamma_c(x,y) \tag{59}$$

and $\qquad \Gamma_c(x,x) = 1 \qquad , \qquad \Gamma_c(x,y)\,\Gamma_c(y,z) = \Gamma_c(x,z) \qquad (6)$

The question is now what the exact form of the operator Γ_c is. This, of course, depends on the exact form of the equations of motion for $v(x)$. To get some insight into what kind of systems have evolution operators with the properties (58), (59), (60) we look at a simpler problem.

For a set of paths c, we investigate the nature of objects $U_c(y,x)$ with the properties

1) $U_c(y,x)$ is an element of G associated with an oriented path c going from x to y in Minkowski space.

2) $U_c(y,x)$ transforms with respect to a space-time dependent transformation $g(x) \in G$ as

$$U'_c(y,x) = g(y)\, U_c(y,x)\, g(x)^{-1} \qquad (61)$$

3) $U_c(y,x)$ satisfies the group property

$$U_c(y,z)\, U_c(z,x) = U_c(y,x) \qquad (62)$$

for point x, y, z on the curve c.

$U_c(y,x)$ is thus a simpler object than $\Gamma_c(y,x)$ since it is an element of G, and therefore simply an $M \times M$-matrix in our case rather than a general operator.

Let the path $c(\varepsilon)$ be parametrized by a parameter ε. Since $U_c(x,x) = \mathbb{1}$ we have $\left(c : [a,b] \to \mathbb{R}^4 \right)$

$$U_c(x(\varepsilon+d\varepsilon), x(\varepsilon)) = U_c(x(\varepsilon), x(\varepsilon))$$

$$+ \sum_\mu \frac{\partial U_c}{\partial x^\mu} \frac{dc^\mu}{d\varepsilon} d\varepsilon + \ldots$$

$$= \mathbb{1} + \sum_\mu W_\mu(c(\varepsilon)) \frac{dc^\mu}{d\varepsilon} d\varepsilon + \ldots \qquad (63)$$

where we defined

$$W_\mu(x) = \frac{\partial U_c(x,y)}{\partial x^\mu}\bigg|_{x=y} \qquad (64)$$

Technically speaking this is the derivative of a group element in $U_c(x,x) = \mathbb{1}$ and hence an element of the Lie-algebra of G (we assume continuous groups).
Due to the group property (62) we further have

$$U_c(x(e+de), x(a)) = U_c(x(e+de), x(e))\, U_c(x(e), x(a))$$

$$= \left\{ \mathbb{1} + \sum_\mu W_\mu \frac{dc^\mu}{de} de + \ldots \right\} U_c(x(e), x(a)) \qquad (64)$$

where $c(a)$ is the beginning of the curve $c(t)$. So

$$U_c(x(e+de), x(a)) - U_c(x(e), x(a)) = \sum_\mu W_\mu \frac{dc^\mu}{de} de \; U_c(x(e), x(a)) \qquad (65)$$

or equivalently

$$\frac{d\, U_c(x(e), x(a))}{de} = W(e)\, U_c(x(e), x(a)) \qquad (66)$$

where we used the abbreviation

$$W(e) = \sum_\mu W_\mu(c(e)) \frac{dc^\mu}{de} \qquad (67)$$

We can now integrate (66) to obtain

$$U_c(x(e), x(a)) = \mathbb{1} + \int_a^e de_1\, W(e_1)\, U_c(x(e_1), x(a)) \qquad (68)$$

Inserting the left hand side of (68) back into the right hand side then gives

$$U_c(x(e), x(a)) = \mathbb{1} + \int_a^e de_1\, W(e_1)$$

$$+ \int_a^e de_1 \int_a^{e_1} de_2\, W(e_1) W(e_2)\, U_c(x(e_2), x(a)) \qquad (69)$$

Iterating even further then gives the equation

$$U_c(x(e), x(a)) =$$

$$\mathbb{1} + \sum_{n=1}^{\infty} \int_a^{e} de_1 \int_a^{e_1} de_2 \ldots \int_a^{e_{n-1}} de_n \; W(e_1) \ldots W(e_n) \qquad (70)$$

We can now define the path-ordered product of n - matrices $W(e_j)$ by

$$\mathcal{T}_c \left(W(e_1) \ldots W(e_n) \right)$$

$$\equiv \sum_P \Theta(e_{P(1)}, \ldots, e_{P(n)}) \, W(e_{P(1)}) \ldots W(e_{P(n)}) \qquad (71)$$

where

$$\Theta(e_1, \ldots, e_n) = \begin{cases} 1 & \text{if } e_1 > e_2 > \ldots > e_n \\ 0 & \text{otherwise} \end{cases} \qquad (72)$$

and P a permutation of the labels $(1, \ldots, n)$.
So the operation $\mathcal{T}_c$ in (71) always orders the matrix $W(e)$ which is "latest" as measured along the path to the left in the product. Then Eq. (70) becomes

$$U_c(x(b), x(a))$$

$$= \mathbb{1} + \sum_{n=1}^{\infty} \int_a^b de_1 \ldots \int_a^b de_n \; \Theta(e_1, \ldots, e_n) \, W(e_1) \ldots W(e_n)$$

$$= \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_P \int_a^b de_1 \ldots \int_a^b de_n \; \Theta(e_{P(1)}, \ldots, e_{P(n)}) \, W(e_{P(1)}) \ldots W(e_{P(n)})$$

$$= \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_a^b de_1 \ldots \int_a^b de_n \; \mathcal{T}_c \left(W(e_1) \ldots W(e_n) \right)$$

$$\equiv \mathcal{T}_c \exp \left(\int_a^b de \; W(e) \right) \qquad (73)$$

where the last line defines the path-ordered exponential. With (67) we can thus write

$$U_c(x(b), x(a)) = \mathcal{T}_c \exp \left(\int_a^b de \sum_\mu W_\mu(c(e)) \frac{dc^\mu}{de} de \right)$$

$$= \mathcal{T}_c \exp \left(\int_c \sum_\mu W_\mu(x) \, dx^\mu \right) \qquad (74)$$

The matrix $U_C(x(b), x(a))$ is thus obtained from a path-ordered exponent of a line-integral of a matrix-valued 1-form

$$W = \sum_\mu W_\mu(x) dx^\mu \tag{75}$$

along a curve C. The one-form W is often called the connection one-form.
The condition (61) puts a condition on the one-form W. We have

$$U'_C(x+dx, x) = g(x+dx)\, U_C(x+dx, x)\, g^{-1}(x) \tag{76}$$

and therefore from (63) we see that

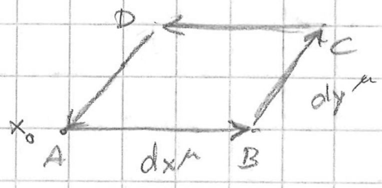
$$\begin{aligned} &\mathbb{1} + \sum_\mu W'_\mu(x) dx^\mu + \dots \\ &= \left\{ g(x) + \sum_\nu \frac{\partial g}{\partial x^\nu} dx^\nu + \dots \right\} \left\{ \mathbb{1} + \sum_\mu W_\mu dx^\mu \right\} g^{-1}(x) \\ &= \mathbb{1} + \sum_\mu \left\{ \frac{\partial g}{\partial x^\mu} g^{-1}(x) + g(x) W_\mu(x) g^{-1}(x) \right\} dx^\mu + \dots \end{aligned} \tag{77}$$

and we find

$$W'_\mu(x) = g(x) W_\mu(x) g^{-1}(x) + \frac{\partial g}{\partial x^\mu} g^{-1}(x) \tag{78}$$

According to our discussion above the one-form W_μ describes the presence of a force-field. Then the force field W'_μ given by (78) represents a physically equivalent force-field, i.e. indistinguishable in its predictions. One says that W and W' are in the same gauge-class.
As a side-remark we note that the transformation law (78) is identical to those of the Christoffel symbols (see (92), Lecture Notes 5). This is no coincidence, the Christoffel symbols describe the connection 1-form of the group of all coordinate transformations.

Let us now derive a quantity that measures the strength of the field W_μ. This will be called the gauge curvature. To do this we consider a closed path $A \to B \to C \to D \to A$ as follows



(79)

Because of the group property (62) we have

$$U_c(A,A) = U_c(A,D)\,U_c(D,C)\,U_c(C,B)\,U_c(B,A)$$

$$= U_c(x_0, x_0+dy)\,U_c(x_0+dy, x_0+dx+dy)\,U_c(x_0+dx+dy, x_0+dx)\,U_c(x_0+dx, x_0)$$

(80)

Using (64) we then have

$$U_c(x_0, x_0+dy) = U_c(x_0+dy-dy, x_0+dy)$$

$$= \mathbb{1} - \sum_\mu W_\mu(x_0+dy)\,dy^\mu + \ldots$$

$$U_c(x_0+dy, x_0+dx+dy) = U_c(x_0+dx+dy-dx, x_0+dx+dy)$$

$$= \mathbb{1} - \sum_\nu W_\nu(x_0+dx+dy)\,dx^\nu + \ldots$$

$$U_c(x_0+dx+dy, x_0+dx) = \mathbb{1} + \sum_\mu W_\mu(x_0+dx)\,dy^\mu + \ldots$$

$$U_c(x_0+dx, x_0) = \mathbb{1} + \sum_\nu W_\nu(x_0)\,dx^\nu + \ldots$$

Multiplying all these expressions in (80) then gives

$$U_c(A,A) = \mathbb{1} + \sum_\mu \left(W_\mu(x_0+dx) - W_\mu(x_0+dy)\right)dy^\mu$$

$$- \sum_\nu \left(W_\nu(x_0+dx+dy) - W_\nu(x_0)\right)dx^\nu +$$

$$+ \sum_{\mu\nu} \{W_\mu(x_0)W_\nu(x_0) - W_\mu(x_0)W_\nu(x_0)$$

$$- W_\nu(x_0)W_\mu(x_0) + W_\mu(x_0)W_\nu(x_0)\}\,dy^\mu dx^\nu + O\left((dx)^2, (dy)^2\right)$$

$$= \mathbb{1} + \left(\frac{\partial W_\mu}{\partial x^\nu} - \frac{\partial W_\nu}{\partial x^\mu} - W_\nu W_\mu + W_\mu W_\nu\right) dy^\mu dx^\nu + \ldots$$

$$= \mathbb{1} + G_{\nu\mu}\,dx^\nu dy^\mu + \ldots \qquad (81)$$

where we defined

$$G_{\nu\mu} = \frac{\partial W_\mu}{\partial x^\nu} - \frac{\partial W_\nu}{\partial x^\mu} - [W_\nu, W_\mu] \tag{82}$$

where $[A, B] = AB - BA$ is the commutator of the matrices A and B.

The quantity $G_{\nu\mu}$ is called the gauge-curvature tensor. The deviation of $U_c(x_0, x_0)$ from unity for a closed path c measures the strength of the field W_μ. Lets see how the curvature tensor looks like for another element W'_μ of the gauge class. To do this we use (78) and the relation

$$g g^{-1} = 1 \Rightarrow 0 = \frac{\partial}{\partial x^\nu}\{g g^{-1}\} = g \frac{\partial g^{-1}}{\partial x^\nu} + \frac{\partial g}{\partial x^\nu} g^{-1}$$

$$\Rightarrow \frac{\partial g^{-1}}{\partial x^\nu} = -g^{-1} \frac{\partial g}{\partial x^\nu} g^{-1} \tag{83}$$

We then have that

$$G'_{\nu\mu} = \frac{\partial W'_\mu}{\partial x^\nu} - \frac{\partial W'_\nu}{\partial x^\mu} - [W'_\nu, W'_\mu] \tag{84}$$

where from (78)

$$\begin{aligned}
[W'_\nu, W'_\mu] &= \left(g W_\nu g^{-1} + \frac{\partial g}{\partial x^\nu} g^{-1}\right)\left(g W_\mu g^{-1} + \frac{\partial g}{\partial x^\mu} g^{-1}\right) - (\mu \leftrightarrow \nu) \\
&= g W_\nu W_\mu g^{-1} + \frac{\partial g}{\partial x^\nu} W_\mu g^{-1} + g W_\nu g^{-1} \frac{\partial g}{\partial x^\mu} g^{-1} + \frac{\partial g}{\partial x^\nu} g^{-1} \frac{\partial g}{\partial x^\mu} g^{-1} - (\mu \leftrightarrow \nu) \\
&= g W_\nu W_\mu g^{-1} + \frac{\partial g}{\partial x^\nu} W_\mu g^{-1} - g W_\nu \frac{\partial g^{-1}}{\partial x^\mu} - \frac{\partial g}{\partial x^\nu} \frac{\partial g^{-1}}{\partial x^\mu} - (\mu \leftrightarrow \nu)
\end{aligned} \tag{85}$$

and

$$\begin{aligned}
\frac{\partial W'_\mu}{\partial x^\nu} - \frac{\partial W'_\nu}{\partial x^\mu} &= \frac{\partial}{\partial x^\nu}\left(g W_\mu g^{-1} + \frac{\partial g}{\partial x^\mu} g^{-1}\right) - (\mu \leftrightarrow \nu) \\
&= \frac{\partial g}{\partial x^\nu} W_\mu g^{-1} + g \frac{\partial W_\mu}{\partial x^\nu} g^{-1} + g W_\mu \frac{\partial g^{-1}}{\partial x^\nu} + \frac{\partial^2 g}{\partial x^\mu \partial x^\nu} g^{-1} + \frac{\partial g}{\partial x^\mu} \frac{\partial g^{-1}}{\partial x^\nu} - (\mu \leftrightarrow \nu)
\end{aligned} \tag{86}$$

Subtracting (85) from (86) then yields

$$G'_{\nu\mu} = g\left(\frac{\partial W_\mu}{\partial x^\nu} - \frac{\partial W_\nu}{\partial x^\mu} - [W_\nu, W_\mu]\right) g^{-1} = g\, G_{\nu\mu}\, g^{-1} \tag{87}$$

We see that the curvature tensor transforms in a very simple way.

Let us now take a simple case. We take W_μ to be a 1×1-matrix, or simply a scalar. We write

$$W_\mu(x) = i\, A_\mu(x) \qquad (88)$$

where $A_\mu(x)$ is a real function. Since scalar functions commute the path-ordered exponential in (73) and (74) becomes a normal exponential and therefore

$$U_C(x(b), x(a)) = e^{i \int_C \sum_\mu A_\mu(x) dx^\mu} \qquad (89)$$

Since this is an element of the group G we see that a general group element is of the form

$$g(x) = e^{i\Lambda(x)} \qquad (90)$$

where $\Lambda(x)$ is a real function. The group G is therefore the group of unitary 1×1 matrices also known as $U(1)$. This is actually the gauge group of electromagnetism. With expression (90) the transformation (78) becomes

$$i\, A'_\mu(x) = e^{i\Lambda(x)} A_\mu(x)\, e^{-i\Lambda(x)} + \left(\frac{\partial}{\partial x^\mu} e^{i\Lambda(x)}\right) e^{-i\Lambda(x)}$$

$$\Rightarrow \quad A'_\mu(x) = A_\mu(x) + \frac{\partial \Lambda}{\partial x^\mu} \qquad (91)$$

or in 1-form notation

$$A' = A + d\Lambda \qquad (92)$$

This is exactly the gauge transformation of the vector potential discussed before Eq. (25) in Lecture Notes 9. The gauge-curvature tensor $G_{\mu\nu}$ of (82) for this group becomes using (88)

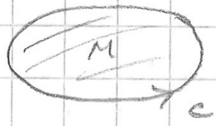
$$G_{\mu\nu} = \frac{\partial}{\partial x^\mu} i A_\nu(x) - \frac{\partial}{\partial x^\nu} i A_\mu = i\, F_{\mu\nu} \qquad (93)$$

where

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \qquad (94)$$

This is the electromagnetic field tensor containing the electric and magnetic fields. For a closed loop c we have that

$$U_c(x,x) = e^{i\int_c \sum_\mu A_\mu dx^\mu} = e^{i\int_M dA} = e^{i\int_M F} \qquad (95)$$



where we integrate the 2-form F over the interior M of the region enclosed by the loop c. As described in (128) of Lecture Notes 13 this is just equal to the magnetic flux when M is an equal-time surface.

In the electromagnetic case it is also easy to check the transformation law (61). From (89) and (91) we have

$$\begin{aligned}
U_c'(x(b), x(a)) &= e^{i\int_c \sum_\mu A'_\mu dx^\mu} \\
&= e^{i\int_c \sum_\mu (A_\mu + \frac{\partial \Lambda}{\partial x^\mu}) dx^\mu} = e^{i\int_c \sum_\mu A_\mu dx^\mu} \, e^{i(\Lambda(x(b)) - \Lambda(x(a)))} \\
&= e^{i\Lambda(x(b))} \, U_c(x(b), x(a)) \, e^{-i\Lambda(x(a))} \\
&= g(x(b)) \, U_c(x(b), x(a)) \, g^{-1}(x(a))
\end{aligned} \qquad (96)$$

We have thus been able to derive some explicit forms for the transformation functions $U_c(y,x)$ along curves c. We have, however, not related those to the evolution operators $\Gamma_c(y,x)$ of Eq. (54). To determine $\Gamma_c(y,x)$ we have to know something about the equations of motion the vector $v(x)$ in (54), and this depends on the system at hand. From our analysis above we, however, realize that $(v(x), W_\mu(x))$ should be in the same physical class as $(g(x)v(x), W_\mu'(x))$, i.e.

$$[v, W_\mu] = [gv, \, g W_\mu g^{-1} + \frac{\partial g}{\partial x^\mu} g^{-1}] \qquad (97)$$

where $[\ ,\ ]$ denotes an equivalence class of vectors v and force fields W_μ. More physically speaking

If (v, W_μ) denotes a particle exposed to a field W_μ then gv describes a physically equivalent solution exposed to the equivalent force-field $W'_\mu = g W_\mu g^{-1} + \frac{\partial g}{\partial x^\mu} g^{-1}$.

We therefore have to find equations of motion for which both sides of Eq. (97) are a solution for an arbitrary choice of group elements $g(x)$. If the equation of motion of the vector $v(x)$ is a differential equation then we have to do this by defining gauge-invariant derivatives. It is readily seen that

$$\left(\frac{\partial}{\partial x^\mu} - W'_\mu(x)\right) g(x) v(x) = \frac{\partial g}{\partial x^\mu} v(x) + g(x) \frac{\partial v}{\partial x^\mu}$$

$$- \left[g W_\mu g^{-1} + \frac{\partial g}{\partial x^\mu} g^{-1}\right] g v = g(x) \left(\frac{\partial}{\partial x^\mu} - W_\mu(x)\right) v(x) \qquad (98)$$

Therefore the combination

$$D_\mu \equiv \frac{\partial}{\partial x^\mu} - W_\mu(x) \qquad (99)$$

has the property that

$$D'_\mu (gv) = g D_\mu v \qquad (100)$$

where $D'_\mu = \frac{\partial}{\partial x^\mu} - W'_\mu(x)$. The operator D_μ is called the gauge-covariant derivative. In general differential equations with the property that both sides of (97) are solutions can be constructed by replacing all derivatives by gauge-covariant ones. A thorough discussion of these issues drives us a bit to far away from the core of this course and is the topic of what is called Yang-Mills theory. Let us, however, give one relevant example for the case of electromagnetism. Take again $W_\mu = i A_\mu$ such that

$$D_\mu = \frac{\partial}{\partial x^\mu} - i A_\mu(x) \qquad (101)$$

and the group is $U(1)$ with $g(x) = e^{i\Lambda(x)}$
We take $v(x)$ to be a complex scalar function. If v satisfies the differential equation

$$D_\mu D^\mu v - m^2 v = 0 \qquad (102)$$

or more explicitly

$$\left\{ \sum_\mu \left(\frac{\partial}{\partial x^\mu} - iA_\mu\right)\left(\frac{\partial}{\partial x^\mu} - iA_\nu\right)\eta_{\mu\nu} - m^2 \right\} v = 0 \qquad (103)$$

with $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ the Minkowski metric.
Then we find that if v is a solution then also $v' = g(x)\, v = e^{i\Lambda} v$ is a solution when $A'_\mu = A_\mu + \frac{\partial \Lambda}{\partial x^\mu}$. If we define the current to be

$$J_\mu = v^* D_\mu v - (D_\mu v)^* v \qquad (104)$$

Then the current is gauge-invariant $J' = J$ and if we solve (102) together with the Maxwell equations

$$dF = 0$$
$$\star d \star F = \frac{4\pi}{c} J \qquad (105)$$

we have a theory with invariance (97).
A full explicit solution of (102) + (104) + (105) will yield the evolution operator Γ_c that we started with.
As a final remark we just mention that in electromagnetism we need not solve the scalar equation (102) but rather a 4-component Dirac equation. This is, however, the topic of quantum electrodynamics and would require a completely new course of which the present course is a good starting point.

That's all folks

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