1. Estimates for the Poisson equation

In this section we deal with elliptic regularity in the category of $L^p$ spaces, obviously a natural class of spaces besides Hölder and Campanato spaces. For the Laplace equation these estimates are usually obtained via potential theoretic methods, i.e. by studying the fundamental solution. For systems, however, it has become customary to base both Schauder and $L^p$ theory on Campanato’s technique.

We have seen earlier the $L^2$ regularity theory.

**Theorem 1.1** ($L^2$-regularity theory for linear elliptic equations). Let $u \in W^{1,2}(\Omega)$ be a weak solution to $Lu = f$ with $f \in L^2(\Omega)$. Then $D^2u \in L^2_{\text{loc}}(\Omega)$ and for any $\Omega' \subset \subset \Omega$, we have

$$
\|u\|_{W^{2,2}(\Omega')} \leq C (\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}).
$$

And if $u \in W^{1,2}_0(\Omega)$ and $\Omega$ is smooth, we have using Poincaré inequality and the global result

$$
\|u\|_{W^{2,2}(\Omega)} \leq C \|f\|_{L^2(\Omega)}.
$$

Our aim is to show that if $-\Delta u = f$ with $f \in L^p$ then $u \in W^{2,p}$.

The strategy is to show that the operators $T_{i,k} : f \mapsto D_{ik}u$ are bounded from $L^2$ to $L^2$ and bounded from $L^\infty$ to some weaker space which is the $BMO$-space and use an interpolation theorem to get the full range $(2, \infty)$. As in the proof of the Schauder estimates, we need to collect some useful preliminary lemmas.

**Lemma 1.2.** Let $u$ be a weak solution to $-\Delta u = 0$ in $B(0,2R)$. Then for any $0 < \rho \leq R$, it holds

$$
\int_{B(0,\rho)} u^2(x) \, dx \leq C \left( \frac{\rho}{R} \right)^n \int_{B(0,R)} u^2(x) \, dx,
$$

$$
\int_{B(0,\rho)} (u - u_\rho)^2(x) \, dx \leq C \left( \frac{\rho}{R} \right)^{n+2} \int_{B(0,R)} (u - u_R)^2(x) \, dx,
$$

with $C = C(n)$.

**Proof.** The proof is the same as in the section concerning the Schauder estimates. \qed
Lemma 1.3. Let $u$ be a solution to $-\Delta u = f$ in $B(0, 2R)$ with $f \in L^\infty(B(0, 2R))$. Let $w := D_i u$. Then for $0 < \rho \leq R$, we have

\[
\int_{B(0, \rho)} |Dw - (Dw)_\rho|^2 \, dx \leq C \left( \frac{\rho}{R} \right)^{n+2} \int_{B(0, R)} |Dw - (Dw)_R|^2 \, dx + CR^n \|f\|^2_{L^\infty(B(0, R))}.
\]

Proof. We proceed by decomposing $w = w_1 + w_2$ where $w_1$ is a solution to the homogeneous problem

\[
\begin{cases}
-\Delta w_1 = 0 & \text{in } B(0, R) \\
w_1 = w & \text{on } \partial B(0, R)
\end{cases}
\]

and $w_2$ satisfies in the weak sense ($D_i f$ to be understood in the weak sense)

\[
\begin{cases}
-\Delta w_2 = D_i f & \text{in } B(0, R) \\
w_2 = 0 & \text{on } \partial B(0, R).
\end{cases}
\]

Since for $i = 1, \ldots, n$, $D_i w_1$ belongs to $W^{1,2}$ (follows from the $L^2$ regularity theory) and since $D_i w_1$ solves $-\Delta(D_i w_1) = 0$ (see exercise set 12), using Lemma 1.2 we have

\[
\int_{B(0, \rho)} |D_i w_1 - (D_i w_1)_\rho|^2 \, dx \leq C \left( \frac{\rho}{R} \right)^{n+2} \int_{B(0, R)} |D_i w_1 - (D_i w_1)_R|^2 \, dx.
\]

Summing over $i$, we get

\[
\int_{B(0, \rho)} |Dw_1 - (Dw_1)_\rho|^2 \, dx \leq C \left( \frac{\rho}{R} \right)^{n+2} \int_{B(0, R)} |Dw_1 - (Dw_1)_R|^2 \, dx.
\]
We have

\[ \int_{B(0,\rho)} |Dw - (Dw)_{\rho}|^2 \, dx \leq C \int_{B(0,\rho)} |Dw_1 - (Dw_1)_{\rho}|^2 \, dx \]

\[ + C \int_{B(0,\rho)} |Dw_2 - (Dw_2)_{\rho}|^2 \, dx \]

\[ \leq C \left( \frac{\rho}{R} \right)^{n+2} \int_{B(0,R)} |Dw_1 - (Dw_1)_R|^2 \, dx \]

\[ + C \int_{B(0,\rho)} |Dw_2 - (Dw_2)_{\rho}|^2 \, dx \]

\[ \leq C \left( \frac{\rho}{R} \right)^{n+2} \int_{B(0,R)} |Dw - (Dw)_R|^2 \, dx \]

\[ + C \int_{B(0,R)} |Dw_2|^2 \, dx \]

where we used that \((Dw_2)_R = 0\) since \(w_2 = 0\) on \(\partial B(0, R)\),

\[ \int_{B(0,R)} |Dw_1 - (Dw_1)_R|^2 \, dx = \int_{B(0,R)} |Dw - Dw_2 - (Dw)_R - (Dw_2)_R|^2 \, dx \]

\[ \leq 2 \int_{B(0,R)} |Dw - (Dw)_R|^2 \, dx + 2 \int_{B(0,R)} |Dw_2|^2 \, dx \]

and the fact that the auxiliary function

\[ g(\rho) := \int_{B(0,\rho)} |Dw_2 - (Dw_2)_{\rho}|^2 \, dx \]

is non decreasing because of the minimality property of the mean \((Dw_2)_{\rho}\) (see exercise set 12). Now we have to estimate \(\int_{B(0,R)} |Dw_2|^2 \, dx\).

Since \(w_2 = 0\) on the boundary, we can take \(\varphi = w_2\) as a test function in the weak formulation \(\int_{B(0,R)} Dw_2 \cdot D\varphi \, dx = -\int_{B(0,R)} f D\varphi \, dx\) and
we get
\[
\int_{B(0,R)}|Dw_2|^2 \, dx = -\int_{B(0,R)} fD_2w_2 \, dx \\
\leq \frac{1}{2} \int_{B(0,R)} f^2 \, dx + \frac{1}{2} \int_{B(0,R)} |Dw_2|^2 \, dx \\
\leq C(n)R^n \|f\|_{L^\infty(B(0,R))} + \frac{1}{2} \int_{B(0,R)} |Dw_2|^2 \, dx.
\]
Combining all the estimates, we arrive to the desired result. □

Using the following iteration lemma, we are able to show that if \( f \in L^\infty \) then \( D^2u \) belongs to the Campanato space \( \mathcal{L}^{2,n} \).

**Lemma 1.4.** Let \( G \) be a nonnegative and non-decreasing function satisfying
\[
G(\rho) \leq A \left( \frac{\rho}{R} \right)^\gamma G(R) + BR^\beta, \quad 0 < \rho \leq R \leq R_0
\]
where \( 0 < \beta < \gamma \) and \( A > 0 \). Then, there exists \( C = C(A, \gamma, \beta) \) such that
\[
G(\rho) \leq C \left( \frac{\rho}{R} \right)^\beta (G(R) + BR^\beta), \quad 0 < \rho \leq R \leq R_0.
\]

**Proof.** See exercise set 12. □

**Lemma 1.5.** Let \( f \in L^\infty(B(0,2R)) \) and \( u \in W^{1,2}(B(0,2R)) \) be a weak solution to \(-\Delta u = f\). Then for any \( 0 < \rho \leq R \), we have
\[
\int_{B(0,\rho)} |Dw - (Dw)_\rho|^2 \, dx \leq \\
C \left( \frac{\rho}{R} \right)^n \left( \int_{B(0,R)} |Dw - (Dw)_R|^2 \, dx + R^n \|f\|_{L^\infty(B(0,R))}^2 \right),
\]
where \( C = C(n) \)

**Proof.** By the previous theorem, we have that \( G(r) := \int_{B(0,\rho)} |Dw - (Dw)_\rho|^2 \, dx \) satisfies the assumption of the iteration lemma and the conclusion follows. □

Let \( 0 < \rho \leq R \leq R_0 \). Using the translation invariance of the equation, we have:

**Theorem 1.6.** Let \( Q_0 \) be a cube, \( f \in L^\infty(Q_0) \) and \( u \in W^{1,2}_0(Q_0) \) be a weak solution to \(-\Delta u = f\) in \( Q_0 \). Let \( B(x_0,3R_0) \subset \subset Q_0 \). Then, \( D^2u \in \mathcal{L}^{2,n}(B(x_0,R_0)) \) and
\[
|D^2u|_{\mathcal{L}^{2,n}(B(x_0,R_0))} \leq C \left( \|D^2u\|_{L^2(B(x_0,2R_0))} + \|f\|_{L^\infty(B(x_0,2R_0))} \right).
\]
Proof. By the translation invariance of the equation and Lemma 1.5, we have for $w = D_i u$ and $x \in B(x_0, R_0)$

$$
\int_{B(x, \rho)} |Dw - (Dw)_{x, \rho}|^2 \, dx \leq C \left( \frac{\rho}{R} \right)^n \left( \int_{B(x, R)} |Dw - (Dw)_{x, R}|^2 \, dx + R^n \|f\|_{L^\infty(B(x, R)))}^2 \right)
$$

$$
\leq C\rho^n \left( \frac{1}{R^n} \|Dw\|_{L^2(B(x, R))}^2 + \|f\|_{L^\infty(B(x, R))}^2 \right).
$$

We hence obtain

$$
\int_{B(x, \rho) \cap B(x_0, R_0)} |Dw - (Dw)_{B(x, \rho) \cap B(x_0, R_0)}|^2 \, dx \leq \int_{B(x, \rho)} |Dw - (Dw)_{x, \rho}|^2 \, dx
$$

$$
\leq C\rho^n \left( \frac{1}{R^n} \|Dw\|_{L^2(B(x, R))}^2 + \|f\|_{L^\infty(B(x, R))}^2 \right)
$$

From this we deduce that

$$
|Dw|_{L^{2,n}(B(x_0, R_0))} := \sup_{x \in B(x_0, R_0), 0 < \rho < R_0} \left( \rho^{-n} \int_{B(x, \rho) \cap B(x_0, R_0)} |Dw - (Dw)|^2, dy \right)^{1/2}
$$

$$
\leq C \left( \frac{1}{R_0^n} \|Dw\|_{L^2(B(x_0, R_0))} + \|f\|_{L^\infty(B(x_0, R_0))} \right).
$$

Summing over $i$ we get the desired result. \qed

By reflecting the solution across the boundaries and using that for any solution $u \in W_0^{1,2}(Q)$ of $-\Delta u = f$ we have from the $L^2$-theory

$$
\|u\|_{W^{2,2}(Q)} \leq C \|f\|_{L^2(Q)};
$$

we are able to obtain the global result. We omit the proof (see the book of Wu, Yin and Wang: Elliptic and parabolic equations for details).

Theorem 1.7. Let $f \in L^\infty(Q_0)$ and let $u \in W_0^{1,2}(Q_0)$ be a weak solution to $-\Delta u = f$. Then

$$
\|D^2 u\|_{L^{2,n}(Q_0)} \leq C(n) \|f\|_{L^\infty(Q_0)}.
$$
Definition 1.8 (BMO-space, bounded mean oscillation). The space $\text{BMO}(Q_0)$ consists of all functions $u \in L^1(Q_0)$ such that

$$|u|_{*,Q_0} := \sup_Q \frac{1}{|Q \cap Q_0|} \int_{Q \cap Q_0} |u - u_{Q \cap Q_0}| \, dx < \infty,$$

where the sup is taken over cubes $Q$ with center in $Q_0$ and parallel to $Q_0$. The norm in $\text{BMO}(Q_0)$ is given by

$$\|\cdot\|_{\text{BMO}(Q_0)} := \|\cdot\|_{L^1(Q_0)} + |\cdot|_{*,Q_0}.$$

Proposition 1.9. If $f \in L^2_n(Q_0)$ then $f \in \text{BMO}(Q_0)$ and

$$\|f\|_{\text{BMO}(Q_0)} \leq \|f\|_{L^2_n(Q_0)}.$$

Proof. see exercise set 13. \qed

Theorem 1.10 (Stampacchia’s interpolation theorem). Let $Q_0$ be a cube. Let $1 < q < \infty$. If $T$ is a bounded continuous linear operator from $L^q(Q_0)$ to $L^q(Q_0)$ and continuous from $L^\infty(Q_0)$ to $\text{BMO}(Q_0)$, that is there exist constants $A_q > 0$ and $A_{\text{BMO}} > 0$ such that

$$\|Tf\|_{L^q(Q_0)} \leq A_q \|f\|_{L^q(Q_0)} \quad \forall f \in L^q(Q_0)$$

$$\|Tf\|_{\text{BMO}(Q_0)} \leq A_{\text{BMO}} \|f\|_{L^\infty(Q_0)} \quad \forall f \in L^\infty(Q_0).$$

Then, $T$ is a linear continuous operator from $L^p(Q_0)$ to $L^p(Q_0)$ for any $q < p < \infty$, that is there exists a constant $C = C(n,p,q,A_q,A_{\text{BMO}}) > 0$ such that

$$\|Tf\|_{L^p(Q_0)} \leq C \|f\|_{L^p(Q_0)} \quad \forall f \in L^p(Q_0).$$

The proof of the Stampacchia’s interpolation theorem is postponed later and is based on three fundamental theorems: Marcinkiewicz’s interpolation theorem, Hardy Littlewood maximal theorem and the Fefferman-Stein’s theorem. Another proof using Marcinkiewicz’s theorem and a version of John-Nirenberg lemma characterizing $\text{BMO}$ functions can be found in the book of Giaquinta and Matinazzi.

Combining the $L^2$-Theory estimates, the result of Theorem 1.7, Proposition 1.9 and Stampacchia’s interpolation theorem, we obtain the following.

Theorem 1.11. Let $p \in [2, \infty)$ and $f \in L^p(Q_0)$. Let $u \in W^{1,2}_0(Q_0)$ be a weak solution to $-\Delta u = f$. Then, there exists a constant $C = C(n,p) > 0$ such that

$$\|D^2 u\|_{L^p(Q_0)} \leq C \|f\|_{L^p(Q_0)}.$$

Now to get an estimate of the full-norm $W^{2,p}$ of $u$ in terms of the $L^p$ norm of $f$, we will need the following interpolation theorem. We omit the proof (see the book of Adams and the book of Wu-Yin-Wang).
Theorem 1.12 (Ehrling-Gagliardo-Nirenberg). Let \( \Omega \) be a bounded domain satisfying a uniform inner cone property and \( p \geq 1 \). Then for any \( \varepsilon > 0 \) there exists a constant \( C = C(\varepsilon, p, k, \Omega) > 0 \) such that for any \( u \in W^{k,p}(\Omega) \), we have

\[
\sum_{|\alpha| \leq k-1} \int_\Omega |D^\alpha u|^p \, dx \leq \varepsilon \sum_{|\alpha| = k} \int_\Omega |D^\alpha u|^p \, dx + C \int_\Omega |u|^p \, dx.
\]

We say that \( \Omega \) a bounded domain satisfies a uniform inner cone property if there exists a finite cone \( V \) such that \( \forall x \in \Omega \) there exists a finite cone \( V_x \) with vertex \( x \) and congruent to \( V \) such that \( V_x \subset \Omega \).

Theorem 1.13 (\( L^p \)-regularity estimates). Let \( p \in [2, \infty) \) and \( f \in L^p(Q_0) \), \( u \in W^{1,2}(Q_0) \) be a weak solution to \( -\Delta u = f \). Then \( u \in W^{2,p}(Q_0) \) and there exists a constant \( C = C(n, p) > 0 \) such that

\[
\|u\|_{W^{2,p}(Q_0)} \leq C \left( \|f\|_{L^p(Q_0)} + \|u\|_{L^p(Q_0)} \right).
\]

The \( L^p \) regularity theory is also valid for \( p \in (1, 2) \) but the proof relies on different arguments (see the book of Chen and Wu chapter 3 based on potential theory and interpolation arguments). However the estimates fails for the extremal cases \( p = 1 \) and \( p = \infty \).

Counterexamples for \( p = \infty \) and \( p = 1 \). Failure of the \( L^\infty \) estimate. Let \( D^2 := \{ x \in \mathbb{R}^2 \mid |x| < 1 \} \) and consider the function \( u \) definite in polar coordinate \( u(x) = r^2 \ln(e^2 - 1) \). The function \( u \) solves

\[
\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 4e^{2\theta}
\]

The right hand term is in \( L^\infty \) but \( D^2 u \notin L^\infty \). However \( D^2 u \in \text{BMO} \) (Ex).

Failure of the \( L^1 \) estimate. Let \( D^2 := \{ x \in \mathbb{R}^2 \mid |x| < 1 \} \) and consider the function \( u(x) = \ln(\ln(e|x|^{-1})) \). The function \( u \) solves

\[
\Delta u = -\frac{1}{|x|^2 \ln^2(e|x|^{-1})}
\]

The right hand term belongs to \( L^1(D^2) \) due to \( \int_0^1 \frac{1}{r \ln^2(e^{-1})} \, dr < \infty \) but \( u \notin W^{2,1}(D^2) \). Indeed \( |D^2 u| \geq \frac{\partial^2 u}{\partial r^2} = \frac{\ln(e^{-1})}{r^2 \ln(e^{-1})} \geq \frac{1}{2r^2 \ln(e^{-1})} \)
2. Proof of the Stampacchia interpolation theorem

2.1. Preliminary tools from real analysis.

**Definition 2.1** (The distribution function). Let \( \Omega \) be an open set and \( f : \Omega \mapsto \mathbb{R} \) a measurable function. Given \( t \geq 0 \), we denote 
\[
A_f(t) := \{ x \in \Omega \mid |f(x)| > t \}.
\]

The distribution function of \( f \) is the function \( \lambda_f(t) : [0, +\infty) \mapsto \mathbb{R} \) defined by 
\[
\lambda_f(t) := |A_f(t)|.
\]

We have the following properties that link the distribution function with \( L^p \) norms (Lebesgue integral characterization).

**Proposition 2.2.** For all \( p \in (0, +\infty) \), we have 
\[
\int_\Omega |f|^p \, dx = p \int_0^{+\infty} s^{p-1} \lambda_f(s) \, ds,
\]
and for \( p = \infty \), we have 
\[
\|f\|_{L^\infty(\Omega)} = \inf \{ t \geq 0 \mid \lambda_f(t) = 0 \}.
\]

**Definition 2.3.** Let \( p \in (0, +\infty) \). A measurable function \( f \) is said to be weakly \( p \)-summable or \( f \in L^p_w(\Omega) \) (weak \( L^p \) space), if 
\[
\|f\|_{L^p_w(\Omega)} := \sup_{t > 0} t \lambda_f(t) < \infty.
\]
If \( p = \infty \), we set \( L^\infty_w(\Omega) = L^\infty(\Omega) \).

Notice that \( \|\cdot\|_{L^p_w(\Omega)} \) is not a norm. It is easy to see that \( L^p(\Omega) \subset L^p_w(\Omega) \) and that \( L^p_w(\Omega) \subset L^q(\Omega) \) if \( q < p \) and \( \Omega \) is bounded (Ex).

**Definition 2.4.** Let \( \Omega \) be a measurable set. Let \( T \) be an operator that maps measurable functions on \( \Omega \) into measurable functions on \( \Omega \). \( T \) is said to be quasilinear if there exists a constant \( C > 0 \) such that 
\[
|T(f + g)| \leq C(|T(f)| + |T(g)|).
\]
\( T \) is said to be of weak \((p, q)\)-type with \( p \in [1, \infty] \) and \( q \in [1, \infty] \) if there exists a constant \( A \geq 0 \) such that 
\[
\|Tf\|_{L^q_w(\Omega)} \leq A \|f\|_{L^p(\Omega)}.
\]
That is 
\[
\lambda_{Tf}(s) \leq \left( \frac{A \|f\|_{L^p(\Omega)}}{s} \right)^q, \quad \forall f \in L^p(\Omega) \quad \text{if } q < \infty,
\]
\[
\|Tf\|_{L^\infty_w(\Omega)} \leq A \|f\|_{L^p(\Omega)}, \quad \forall f \in L^p(\Omega), \quad \text{if } q = \infty
\]
Now we have the tools to state the following theorem which will play a central role in the sequel.

**Theorem 2.5** (Marcinkiewicz’s interpolation theorem). Let $T$ be a quasilinear operator both of weak $(p_0, p_0)$-type and weak $(p_1, p_1)$-type, where $1 \leq p_0 < p_1 \leq \infty$. Then $T$ is of strong $(p, p)$-type for all $p_0 < p < p_1$, that is there exists a constant $C > 0$ such that

$$\|Tf\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}, \quad \forall f \in L^p(\Omega).$$

**Proof.** Let $f \in L^p(\Omega)$ with $p_0 < p < p_1$. The idea is to use Proposition 2.2 to compute the $L^p$ norm of $T(f)$ and to control the distribution function of $T(f)$ by splitting $f$ into a ”good” ($f$ small) and a ”bad” ($f$ large) part. We know that there exist positive constants $C_q, A_0, A_1$ such that

$$|T(f + g)| \leq C_q(|T(f)| + |T(g)|)$$

$$\lambda_{Tf}(s) \leq \left(\frac{A_0 \|f\|_{L^{p_0}(\Omega)}}{s}\right)^{p_0}, \quad \forall f \in L^{p_0}(\Omega)$$

$$\lambda_{Tf}(s) \leq \left(\frac{A_1 \|f\|_{L^{p_1}(\Omega)}}{s}\right)^{p_1}, \quad \forall f \in L^{p_1}(\Omega).$$

We fix $s > 0$ and denote $A_{Tf}(s) := \{x \in \Omega \mid |Tf(x)| > s\}$. We split $f$ into two parts $f = f_0 + f_1$ where

$$f_0 := f\chi_{\{x \in \Omega \mid |f(x)| > \frac{s}{2C_q}\}}$$

$$f_1 := f\chi_{\{x \in \Omega \mid |f(x)| \leq \frac{s}{2C_q}\}}.$$

Using the quasilinearity of $T$, we have $|Tf| \leq C_q(|Tf_0| + |Tf_1|)$. It follows that

$$A_{Tf}(s) \subset \left\{x \in \Omega : |Tf_0| > \frac{s}{2C_q}\right\} \cup \left\{x \in \Omega : |Tf_1| > \frac{s}{2C_q}\right\}$$

$$:= A_{Tf_0}\left(\frac{s}{2C_q}\right) \cup A_{Tf_1}\left(\frac{s}{2C_q}\right).$$
Using that \( f_0 \in L^{p_0}(\Omega) \) ( \(|f_0| \leq |f| \) and \( p_0 \leq p \)), \( f_1 \in L^{p_1}(\Omega) \) ( \( f_1 \) is bounded) and the properties of \( T \), we get

\[
\left| A_T f_0 \left( \frac{s}{2C_q} \right) \right| = \lambda_{Tf_0} \leq \left( \frac{A_0 \|f_0\|_{L^{p_0}(\Omega)}}{s} \right)^{p_0}
\]

\[
= \frac{c_0}{s^{p_0}} \int_{\{|f| > s/(2C_qA_1)|\}} |f(x)|^{p_0} \, dx
\]

\[
\left| A_T f_1 \left( \frac{s}{2C_q} \right) \right| = \lambda_{Tf_1} \leq \left( \frac{A_1 \|f_1\|_{L^{p_1}(\Omega)}}{s} \right)^{p_1}
\]

\[
= \frac{c_1}{s^{p_1}} \int_{\{|f| \leq s/(2C_qA_1)|\}} |f(x)|^{p_1} \, dx \quad \text{if} \quad p_1 < \infty
\]

and since \( \|Tf_1\| \leq A_1 \|f_1\|_{L^{\infty}(\Omega)} \leq \frac{s}{2C_q} \) we have that \( A_T f_1 \left( \frac{s}{2C_q} \right) = \emptyset \) if \( p_1 = \infty \). Hence

\[
\lambda_{Tf}(s) = |A_T f(s)| \leq \left| A_T f_0 \left( \frac{s}{2C_q} \right) \right| + \left| A_T f_1 \left( \frac{s}{2C_q} \right) \right|.
\]

We get that

\[
\int_{\Omega} |f|^p \, dx = p \int_0^\infty s^{p-1} \lambda_{Tf}(s) \, ds
\]

\[
\leq p \int_0^\infty s^{p-1} \lambda_{Tf_0}(s/2C_q) \, ds + p \int_0^\infty s^{p-1} \lambda_{Tf_1}(s/2C_q) \, ds
\]

\[
\leq (2C_qA_0)^{p_0} \int_0^\infty s^{p-1-p_0} \, ds \int_{\{|f| > s/(2C_qA_1)|\}} |f|^{p_0} \, dx
\]

\[
+ (2C_qA_1)^{p_1} \int_0^\infty s^{p-1-p_1} \, ds \int_{\{|f| \leq s/(2C_qA_1)|\}} |f|^{p_1} \, dx
\]

\[
\leq (2C_qA_0)^{p_0} \int_{\Omega} |f|^{p_0} \, dx \int_{2C_qA_1|f|}^{2C_qA_1|f|} s^{p-1-p_0} \, ds
\]

\[
+ (2C_qA_1)^{p_1} \int_{\Omega} |f|^{p_1} \, dx \int_{2C_qA_1|f|}^{2C_qA_1|f|} s^{p-1-p_1} \, ds
\]

\[
\leq C(p, p_0, p_1, A_0, A_1, C_q) \|f\|_{L^p(\Omega)}
\]

In the case of \( p_1 = \infty \), it follows quit similarly. \( \square \)
Definition 2.6 (Hardy Littlewood maximal function). For $f \in L^1(Q_0)$, the Hardy-Littlewood maximal function of $f$ is given by

$$Mf(x) := \sup_{x \in Q} \int_{Q \cap Q_0} |f(y)| \, dy, \quad x \in Q_0,$$

where $Q$ is a cube with center inside $Q_0$.

Proposition 2.7. We have the following properties:

1. $M$ is sublinear: $|M(f + g)| \leq |M(f)| + |M(g)|$.
2. If $f \in L^\infty(Q_0)$ then $\|Mf\|_{L^\infty(Q_0)} \leq \|f\|_{L^\infty(Q_0)}$.
3. Due to the Lebesgue’s differentiation theorem $|f(x)| \leq Mf(x)$ a.e $x \in Q_0$.

Theorem 2.8 (Hardy-Littlewood maximal theorem). The operator $M$ is of weak $(1,1)$-type, and

$$\|Mf\|_{L^1(Q_0)} \leq C(n) \|f\|_{L^1(Q_0)}, \quad \forall f \in L^1(Q_0).$$

The proof is omitted (see the book of Giaquinta based on a covering argument). The key result about maximal function is the following.

Corollary 2.9. Let $f \in L^p(Q_0)$ for some $1 < p < \infty$. Then $Mf \in L^p(Q_0)$ and

$$\|Mf\|_{L^p(Q_0)} \leq C(n,p) \|f\|_{L^p(Q_0)}.$$

Proof. See exercise set 13. \qed

In order to prove the interpolation result between $L^p$ and $BMO$, we need to introduce a more relevant function which is the sharp function (also called the maximal mean oscillation function).

Definition 2.10. The sharp function of a function $f \in L^1(Q_0)$ is given by

$$f^\sharp(x) := \sup_{Q} \int_{Q \cap Q_0} |f(y) - f_{Q \cap Q_0}| \, dy, \quad x \in Q_0,$$

where $Q$ has center inside $Q_0$.

We have the following properties.

Proposition 2.11. We have the following properties:

i) $f \in BMO(Q_0)$ if and only if $f^\sharp \in L^\infty(Q_0)$.
ii) If $1 < p < \infty$ and $f \in L^p(Q_0)$ then $f^\sharp \in L^p(Q_0)$ and

$$\|f^\sharp\|_{L^p(Q_0)} \leq C(n,p) \|f\|_{L^p(Q_0)}.$$

Proof. See exercise set 13. \qed

Conversely to ii), we have the following theorem.
Theorem 2.12 (Fefferman-Stein interpolation theorem). Let $f \in L^1(Q_0)$. If $f^2 \in L^p(Q_0)$ for some $1 < p < \infty$, then $f \in L^p(Q_0)$ and
\[
\|f\|_{L^p(Q_0)} \leq C(n, p) \left( \|f^2\|_{L^p(Q_0)} + |Q_0|^{1/p} \int_{Q_0} |f| \, dx \right).
\]

Now we have the tools to prove Stampacchia’s interpolation theorem.

Proof of Theorem 1.10. Let $u \in L^p(Q_0)$ and let $T$ be a linear operator. We define the operator $F(u) := (Tu)^2$. We are going to show that $F$ satisfies the hypothesis of Marcinkiewicz’s interpolation theorem. Since $T$ is linear and $\sharp$ is sublinear, we have that $F$ is sublinear. Now let us show that $F$ is of strong $(\infty, \infty)$-type and of strong $(q, q)$-type. We have using the properties of the sharp function and the properties of $T$ that
\[
\|Fu\|_{L^q(Q_0)} = \|(Tu)^2\|_{L^q(Q_0)} \leq C(n, q) \|Tu\|_{L^q(Q_0)} \leq C(n, q) A_q \|u\|_{L^q(Q_0)} ,
\]
\[
\|F(u)\|_{L^\infty(Q_0)} = \|(Tu)^2\|_{L^\infty(Q_0)} = |Tu|_{*, Q_0} \leq A_{BMO} \|u\|_{L^\infty(Q_0)}.
\]

Applying Marcinkiewicz’s interpolation theorem to $F$ we get that there exists a constant $C = C(q, n, A_q, A_{BMO}) > 0$ such that for all $u \in L^p(Q_0)$ with $p \in (q, \infty)$, we have
\[
\|(Tu)^2\|_{L^p(Q_0)} = \|Fu\|_{L^p(Q_0)} \leq C \|u\|_{L^p(Q_0)}.
\]

Using now the Fefferman-Stein’s theorem and a Hölder inequality we get
\[
\|Tu\|_{L^p(Q_0)} \leq C \left( \|(Tu)^2\|_{L^q(Q_0)} + |Q_0|^{1/p} \int_{Q_0} |Tu| \, dx \right)
\]
\[
\leq C \left( \|u\|_{L^p(Q_0)} + |Q_0|^{1/p-1/q} \|Tu\|_{L^q(Q_0)} \right)
\]
\[
\leq C \left( \|u\|_{L^p(Q_0)} + |Q_0|^{1/p-1/q} A_p \|u\|_{L^q(Q_0)} \right)
\]
\[
\leq C \|u\|_{L^p(Q_0)},
\]
with $C = C(n, p, q, A_p, A_{BMO})$.

2.2. Proof of the Fefferman Stein’s theorem. In order to prove the Fefferman-Stein’s theorem, we will need the following Calderón-Zygmund decomposition lemma.

Lemma 2.13. Let $Q$ be an $n$-dimensional cube and let $f$ be a non-negative function in $L^1(Q)$. Fix a parameter $\alpha > 0$ such that $\frac{\int_Q f(x) \, dx}{\mu(Q)} \leq \alpha$. Then there exists a countable family of cubes $\{Q_i\}_{i \in I}$ such that
\[
\text{i) } f(x) \leq \alpha \text{ for a.e. } x \in Q \setminus \bigcup_{i \in I} Q_i.
\]
ii) \( \alpha < \int_{Q_i} f \, dx \leq 2^n \alpha \) for every \( i \in I \).

**Proof.** We divide \( Q \) into \( 2^n \) congruent and equal subcubes \( Q' \). There are only two possibilities:

1. \( \int_{Q'} f \, dx > \alpha \).
2. \( \int_{Q'} f \, dx \leq \alpha \).

Those cubes which satisfy the first case \( (\int_{Q'} f \, dx > \alpha) \) will belong to our family \( \{Q_i\} \), while the others are similarly divided into subcubes. This process will continue until that the first case appears. Let \( \{Q_i\} \) be the family of subcubes so obtained for which \( \int_{Q_i} f \, dx > \alpha \) and for each \( Q_i \) denote \( \tilde{Q}_i \), the cube whose subdivision gave rise to \( Q_i \), then

\[
\alpha < \int_{Q_i} f \, dx \leq \frac{1}{2^{-n}|Q_i|} \int_{\tilde{Q}_i} f \, dx \leq 2^n \alpha.
\]

Now if \( x \in Q \setminus \bigcup_{i \in I} Q_i \) and is not on the boundary of some \( Q_i \), then it belongs to infinitely many cubes \( P \) in the successive subdivision with \( |P| \to 0 \). Furthermore, for each such \( P \), the second case is valid \( (\int_{Q'} f \, dx \leq \alpha) \). Since \( f \) is integrable, using the Lebesgue’s differentiation theorem, we get

\[
f(x) := \lim_{x \to P, |P| \to 0} \frac{1}{|P|} \int_{P} f(y) \, dy \leq \alpha, \quad \text{a.e. } x \in Q \setminus \bigcup_{i \in I} Q_i.
\]

**Remark 2.14.** The lemma implies that

\[
| \bigcup_{i \in I} Q_i | = \sum_{i \in I} |Q_i| \leq \sum_{i \in I} \frac{1}{\alpha} \int_{Q_i} f(y) \, dy \leq \frac{1}{\alpha} \| f \|_{L^1(Q)}.
\]

If \( f \in L^1(\mathbb{R}^n) \) and \( \alpha \) is any positive constant, then the conclusion of the previous lemma holds true, since we can first subdivide \( \mathbb{R}^n \) into cubes \( Q \) for which we have

\[
\int_{Q} f \, dx \leq \alpha.
\]

Let \( \{Q_i^{\alpha_1}\} \) and \( \{Q_i^{\alpha_2}\} \) be the families in the decomposition of Calderón-Zygmund corresponding to the parameter \( 0 < \alpha_1 < \alpha_2 \). Then each \( Q_i^{\alpha_2} \) is contained in some \( Q_i^{\alpha_1} \).
Proof of the Fefferman-Stein theorem. Let \( f \in L^1(Q_0) \) satisfying \( f^\sharp \in L^p(Q_0) \) for some \( 1 < p < \infty \). Fix \( \alpha > 0 \) such that \( \int_{Q_0} |f| \, dx < \alpha \). Applying the Calderón-Zygmund lemma to \(|f|\), we get that there exists a family of nonoverlapping cubes \( \{Q_\alpha^i\}_{i \in I} \) such that

\[
\alpha < \int_{Q_0^\alpha} |f| \, dx \leq 2^n \alpha
\]

\[
|f(x)| \leq \alpha, \quad \text{a.e. } x \in Q_0 \setminus \bigcup_{i \in I} Q_\alpha^i.
\]

For \( \alpha > \int_{Q_0} |f| \, dx \) set \( \mu(\alpha) := \sum_{i \in I} |Q_\alpha^i| \). We will need the following proposition

**Proposition 2.15.** For any \( \alpha > 0 \) such that

\[
\alpha > \frac{1}{2n+1} \int_{Q_0} |f| \, dx,
\]

we have

\[
\mu(\alpha) \leq \left| \left\{ x \in Q_0, |f^\sharp(x) > \frac{\alpha}{A} \right\} \right| + 2 \frac{\mu(\frac{\alpha}{2n+1})}{A},
\]

where \( A \) is an arbitrary positive number.

**Proof.** Set \( \beta = \frac{\alpha}{2n+1} \). Let \( \{Q_\alpha^i\}_{i \in I} \) and \( \{Q_\beta^j\}_{j \in J} \) be the Calderón-Zygmund family of cubes corresponding to the function \(|f|\) and the parameters \( \alpha \) and \( \beta \) respectively. We can write

\[
\mu(\beta) = \sum_{j \in J} \sum_{i \in I: Q_\alpha^i \subset Q_\beta^j} |Q_\alpha^i|.
\]

Fix \( j \in J \), then for any \( Q_\alpha^i \subset Q_\beta^j \) there are two possibilities:

1. \( Q_\beta^j \subset \{ x \in Q_0; f^\sharp(x) > \alpha/A \} \). In this case

\[
\sum_{i \in I: Q_\alpha^i \subset Q_\beta^j} |Q_\alpha^i| \leq \left| \left\{ x \in Q_\beta^j; f^\sharp(x) > \alpha/A \right\} \right|.
\]

2. \( Q_\beta^j \not\subset \{ x \in Q_0; f^\sharp(x) > \alpha/A \} \). There exists \( y \in Q_\beta^j \) such that \( f^\sharp(y) \leq \alpha/A \). By the definition of \( f^\sharp \), we have

\[
\int_{Q_\beta^j} |f(x) - f_{Q_\beta^j}| \, dx \leq \alpha/A.
\]
Using that \( \int_{Q^\beta_j} |f| \, dx \leq 2^n \beta = \alpha/2 \) and \( \int_{Q^\alpha_i} |f| \, dx > \alpha \), we have in this case

\[
\int_{Q^\alpha_i} |f(x) - f_{Q^\beta_j}| \, dx \geq \int_{Q^\alpha_i} |f| \, dx - \int_{Q^\beta_j} |f| \, dx \geq \alpha - \alpha/2 = \alpha/2,
\]

that is \( \int_{Q^\alpha_i} |f(x) - f_{Q^\beta_j}| \, dx \geq \alpha/2 |Q^\alpha_i| \).

Hence in the second case

\[
\sum_{i \in I : Q^\alpha_i \subset Q^\beta_j} |Q^\alpha_i| \leq \frac{2}{\alpha} \sum_{i \in I : Q^\alpha_i \subset Q^\beta_j} \int_{Q^\alpha_i} |f(x) - f_{Q^\beta_j}| \, dx \\
\leq \frac{2}{\alpha} \int_{Q^\beta_j} |f(x) - f_{Q^\beta_j}| \, dx \leq \frac{2}{A} |Q^\beta_j|.
\]

Combining the both cases, we have

\[
\sum_{i \in I : Q^\alpha_i \subset Q^\beta_j} |Q^\alpha_i| \leq \left| \{ x \in Q^\beta_j ; \ f^2(x) > \alpha/A \} \right| + \frac{2}{A} |Q^\beta_j|.
\]

Summing over \( j \in J \), we get

\[
\mu(\alpha) \leq \sum_{j \in J} \left( \left| \{ x \in Q^\beta_j ; \ f^2(x) > \alpha/A \} \right| + \frac{2}{A} |Q^\beta_j| \right) \\
\leq \left| \{ x \in Q_0 ; \ f^2(x) > \alpha/A \} \right| + \frac{2}{A} \mu \left( \frac{\alpha}{2^{n+1}} \right).
\]

Now using the Lebesgue integral characterization (Proposition 2.2), we have

\[
\|f\|_{L^p(Q_0)}^p = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) \, d\alpha.
\]

Since \( |f| \leq \alpha \) for a.e \( x \in Q_0 \cup \bigcup_{i \in I} Q^\alpha_i \), we have

\[
\lambda_f(\alpha) = \left| \{ x \in Q_0 ; \ |f| > \alpha \} \right| \leq \sum_i |Q^\alpha_i| = \mu(\alpha).
\]

It follows that

\[
\int_{Q_0} |f|^p \, dx \leq p \int_0^\infty \alpha^{p-1} \mu(\alpha) \, d\alpha.
\]
We fix $s > 2^{n+1} \int_{Q_0} |f| \, dx$ and consider

$$I_s := p \int_0^s \alpha^{p-1} \mu(\alpha) \, d\alpha.$$ 

Using that $\mu(\alpha) \leq |Q_0|$ and Proposition 2.15, we have

$$I_s = p \int_0^{2^{n+1}} \int_{Q_0} |f| \, dx \alpha^{p-1} \mu(\alpha) \, d\alpha + p \int_{2^{n+1}}^s \int_{Q_0} |f| \, dx \alpha^{p-1} \mu(\alpha) \, d\alpha$$

$$\leq |Q_0| \left(2^{n+1} \int_{Q_0} |f| \, dx\right)^p$$

$$+ p \int_0^\infty \left\{|x \in Q_0, |f^\sharp(x)| > \frac{\alpha}{A}\right\} \alpha^{p-1} \, d\alpha$$

$$+ \frac{2p}{A} \int_0^s \alpha^{p-1} \mu\left(\frac{\alpha}{2^{n+1}}\right) \, d\alpha$$

$$\leq |Q_0| \left(2^{n+1} \int_{Q_0} |f| \, dx\right)^p + A^p \|f^\sharp\|^p_{L^p(Q_0)}$$

$$+ \frac{2}{A} 2^{(n+1)p} I_s.$$ 

Choosing $A = 4 \cdot 2^{(n+1)p}$, we get that

$$I_s \leq 2|Q_0| \left(2^{n+1} \int_{Q_0} |f| \, dx\right)^p + 2A^p \|f^\sharp\|^p_{L^p(Q_0)}.$$ 

It follows that

$$p \int_0^s \alpha^{p-1} \lambda_f(\alpha) \, d\alpha \leq I_s$$

$$\leq 2|Q_0| \left(2^{n+1} \int_{Q_0} |f| \, dx\right)^p + 2A^p \|f^\sharp\|^p_{L^p(Q_0)}.$$ 

and letting $s \to \infty$ we get the desired result of Fefferman-Stein.