FYS4120: Adv. ED, Ex. 5, 23.10.2013

1. $\underline{e^i} = g^{is} e_s \underset{\substack{\uparrow \\ \text{metric is} \\ \text{invariant}}}{=} O_{ij} O_{sr} g^{jr} e_s = O_{ij} g^{jr} \underbrace{e_s O_{sr}}_{= f_r} = \underline{O_{ij} f^j}.$

2. Eq. (4): (LHS)

$$\star\left(\sum_{j_1=1}^{n} O_{i_1 j_1} f^{j_1} \wedge \cdots \wedge \sum_{j_p=1}^{n} O_{i_p j_p} f^{j_p} \right)$$

$$= \sum_{j_1,\dots,j_p=1}^{n} O_{i_1 j_1} \cdots O_{i_p j_p} \star (f^{j_1} \wedge \cdots \wedge f^{j_p})$$

$$= \sum_{j_1<\cdots<j_p} \sum_{\sigma} (-1)^\sigma O_{i_1 \sigma(j_1)} \cdots O_{i_p \sigma(j_p)} \epsilon_{j_1 \cdots j_n} \overbrace{\lambda_{j_1} \cdots \lambda_{j_p}}^{\text{metric components}} (f^{j_{p+1}} \wedge \cdots \wedge f^{j_n})$$

$$= \sum_{j_{p+1}<\cdots<j_n} |O|_{(i_1 \cdots i_p, j_1 \cdots j_p)} \epsilon_{j_1 \cdots j_n} \lambda_{j_1} \cdots \lambda_{j_p} (f^{j_{p+1}} \wedge \cdots \wedge f^{j_n}). \quad (*)$$

one-to-one relation between $(j_1 < \cdots < j_p)$ and $(j_{p+1} < \cdots < j_n)$
(see Eqs. (3.100) and (3.101) in the notes.)

Eq. (4): (RHS)

$$\epsilon_{i_1 \cdots i_n} \lambda_{i_1} \cdots \lambda_{i_p} \sum_{j_{p+1}=1}^{n} O_{i_{p+1} j_{p+1}} f^{j_{p+1}} \wedge \cdots \wedge \sum_{j_n=1}^{n} O_{i_n j_n} f^{j_n}$$

$$= \epsilon_{i_1 \cdots i_n} \lambda_{i_1} \cdots \lambda_{i_p} \sum_{j_{p+1}<\cdots<j_n} \sum_{\sigma} (-1)^\sigma O_{i_{p+1} \sigma(j_{p+1})} \cdots O_{i_n \sigma(j_n)} \overbrace{(f^{j_{p+1}} \wedge \cdots \wedge f^{j_n})}^{\text{antisymmetric}}$$

$$= \epsilon_{i_1 \cdots i_n} \lambda_{i_1} \cdots \lambda_{i_p} \sum_{j_{p+1}<\cdots<j_n} |O|_{(i_{p+1} \cdots i_n, j_{p+1} \cdots j_n)} (f^{j_{p+1}} \wedge \cdots \wedge f^{j_n}). \quad (**)$$

Now comparing $(*)$ and $(**)$ gives

$$\underline{|O|_{(i_1 \cdots i_p, j_1 \cdots j_p)} = \lambda_{i_1} \cdots \lambda_{i_p} \lambda_{j_1} \cdots \lambda_{j_p} \delta^{i_1 \cdots i_n}_{j_1 \cdots j_n} |O|_{(i_{p+1} \cdots i_n, j_{p+1} \cdots j_n)}}.$$

3. The given Lorentz transformation is

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}.$$

For the given pair we find

$$\underline{|\Lambda|_{(134,134)}} = \begin{vmatrix} \gamma & 0 & -\beta\gamma \\ 0 & \cos\theta & 0 \\ -\beta\gamma & 0 & \gamma \end{vmatrix} = \cdots = \overbrace{\gamma^2(1-\beta^2)}^{=1} \cos\theta = \underline{\cos\theta},$$

$\underline{\underline{|\Lambda|_{(2,2)} = \cos\theta}}$. Indeed, the values are the same.

4. $\underline{(\tilde{O})_{pq}} = ((\Pi_I)^T O \Pi_J)_{pq} = \delta_{i_p r} O_{rs} \delta_{s j_q} = \underline{O_{i_p j_q}}$,

$\underline{\det(\Pi_J)} = |e_{j_1} \cdots e_{j_n}| = \epsilon_{j_1 \cdots j_n} \underbrace{|e_1 \cdots e_n|}_{=1} = \underline{\epsilon_{j_1 \cdots j_n}}$,

$\underline{\det(\tilde{O})} = \det((\Pi_I)^T O \Pi_J) = \det((\Pi_I)^T) \det(O) \det(\Pi_J)$

$= \det(\Pi_I) \det(O) \det(\Pi_J) = \epsilon_{i_1 \cdots i_n} \epsilon_{j_1 \cdots j_n} \det(O)$

$= \underline{\delta^{i_1 \cdots i_n}_{j_1 \cdots j_n} \det(O)}$,

$\underline{\tilde{\lambda}_J} = \tilde{O}^T \tilde{\lambda}_I \tilde{O} = ((\Pi_I)^T O \Pi_J)^T (\Pi_I^T \lambda \Pi_I) ((\Pi_I)^T O \Pi_J)$

$= (\Pi_J)^T O^T \underbrace{\Pi_I (\Pi_I)^T}_{=\mathbb{1}} \lambda \underbrace{\Pi_I (\Pi_I)^T}_{=\mathbb{1}} O \Pi_J$

$= \Pi_J^T \underbrace{O^T \lambda O}_{=\lambda} \Pi_J = \Pi_J^T \lambda \Pi_J = \underline{\tilde{\lambda}_J} = \underline{\begin{pmatrix} \tilde{\lambda}_{J,1} & 0 \\ 0 & \tilde{\lambda}_{J,2} \end{pmatrix}}$.

5. The trick is to calculate the determinant of R in two ways:

$\underline{\det(R)} = \det\left\{ \begin{pmatrix} A^T & C^T \\ 0 & 1 \end{pmatrix} \tilde{\lambda}_I \tilde{O} \right\} = \underline{\det(A) \det(\tilde{\lambda}_{I,1}) \det(\tilde{\lambda}_{I,2}) \det(\tilde{O})}$. $(\square)$

Explicitly,

$R = \begin{pmatrix} A^T & C^T \\ 0 & 1 \end{pmatrix} \tilde{\lambda}_I \tilde{O} = \begin{pmatrix} A^T & C^T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\lambda}_{I,1} & 0 \\ 0 & \tilde{\lambda}_{I,2} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$= \begin{pmatrix} A^T & C^T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\lambda}_{I,1} A & \tilde{\lambda}_{I,1} B \\ \tilde{\lambda}_{I,2} C & \tilde{\lambda}_{I,2} D \end{pmatrix}$

$= \begin{pmatrix} A^T \tilde{\lambda}_{I,1} A + C^T \tilde{\lambda}_{I,2} C & A^T \tilde{\lambda}_{I,1} B + C^T \tilde{\lambda}_{I,2} D \\ \tilde{\lambda}_{I,2} C & \tilde{\lambda}_{I,2} D \end{pmatrix}$

$= \begin{pmatrix} \tilde{\lambda}_{J,1} & 0 \\ \tilde{\lambda}_{I,2} C & \tilde{\lambda}_{I,2} D \end{pmatrix} \Rightarrow \underline{\det(R) = \det(\tilde{\lambda}_{J,1}) \det(\tilde{\lambda}_{I,2}) \det(D)}$

$(\square\square)$

Put (I) and (II) together to obtain

$$\underline{\det(D)\det(\tilde{\lambda}_{I,1}) = \det(A)\det(\tilde{\lambda}_{I,1})\underbrace{\det(\tilde{O})}_{\text{Prob. 4}}.}$$

Then fill in the quantities

$$|O|_{(i_{p+1}\cdots i_n,\, j_{p+1}\cdots j_n)}\, \lambda_{j_1}\cdots\lambda_{j_p}$$

$$= |O|_{(i_1\cdots i_n,\, j_1\cdots j_n)}\, \lambda_{i_1}\cdots\lambda_{i_p}\, \delta^{i_1\cdots i_n}_{j_1\cdots j_n}\, \overbrace{\det(O)}^{=1}$$

$$\Rightarrow |O|_{(i_1\cdots i_p,\, j_1\cdots j_p)} = \lambda_{i_1}\cdots\lambda_{i_p}\lambda_{j_1}\cdots\lambda_{j_p}\,\delta^{i_1\cdots i_n}_{j_1\cdots j_n}\,|O|_{(i_{p+1},\cdots i_n,\, j_{p+1}\cdots j_n)}\,.$$

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6. The Hodge star is used in Maxwell's equations:

$$dF = 0 \quad\text{and}\quad \star d \star F = j\,.$$

Maxwell's equations depend on the metric; they give us, for instance, Coulomb's law (which depends on distance squared), and the metric gives us the distance.

Maxwell's equations are said to be covariant under the Lorentz group (actually also under the Poincaré group), so not invariant. This means that both sides of the equation transform in the same representation of the group. For example, F is a 2-form; then $d\star F$ is a 3-form an $\star d \star F$ is a 1-form. Then, as long as j is a 1-form, the Maxwell's equation relating $\star d \star F$ to j is a Lorentz covariant equation.

In this exercise we showed that the Hodge star is invariant under orthogonal transformations, e.g., Lorentz transformations. And because of the above reasoning, this must be the case. (otherwise the "mathematical object" would not be supporting the physical idea of Maxwell's equations)