Pairing properties of a Fermi gas with infinite scattering length

Piotr Magierski
Warsaw University of Technology/University of Washington

Collaborators: Aurel Bulgac – University of Washington (Seattle), Joaquín E. Drut – Ohio State University (Columbus), Gabriel Włazłowski (PhD student) – Warsaw University of Technology
What is the unitary regime?

A gas of interacting fermions is in the unitary regime if the average separation between particles is large compared to their size (range of interaction), but small compared to their scattering length.

\[
\frac{E}{E_{FG}} = 1 + \frac{10}{9\pi} (k_F a) \left[ 1 + \frac{6}{35\pi} (k_F a) (11 - 2\ln 2) + \ldots \right] + \text{pairing}
\]

\[
E_{FG} = \frac{3}{5} \varepsilon_F N \quad \text{Energy of the noninteracting Fermi gas}
\]

\[
n \sim \frac{3}{5} n_0^3 \ll 1 \quad \text{and} \quad n |a|^3 \gg 1
\]

\[\text{i.e. } r_0 \to 0, \ a \to \pm \infty\]

**Nonperturbative Regime**

**System is dilute but strongly interacting!**

**Universality:**

\[E = \xi_0 E_{FG}\]

**At finite temperature:**

\[E(T) = \xi \left( \frac{T}{\varepsilon_F} \right) E_{FG}, \ \xi(0) = \xi_0\]
Dilute neutron matter:

Effective range: \( r_0 \approx 2.8 \text{ fm} \)
Scattering length: \( a \approx -18.5 \text{ fm} \)

Unitary gas:

Effective range: \( r_0 \approx 0 \)
Scattering length: \( a \approx \pm \infty \)

Physical realization eg.:
dilute gas of \(^6\text{Li} \text{ atoms}\)
Expected phases of a two species dilute Fermi system
BCS-BEC crossover

Strong interaction
UNITARY REGIME

BCS Superfluid

Molecular BEC and Atomic+Molecular Superfluids

Characteristic temperature: 
$T_c$ superfluid-normal phase transition

Characteristic temperatures:
$T_c$ superfluid-normal phase transition
$T^*$ break up of Bose molecule
$T^* > T_c$

weak interaction

Superfluid

weak interactions

$\frac{1}{a}$

no 2-body bound state

shallow 2-body bound state

strong interaction

Bose molecule

expected phases

a<0

a>0

weak interaction

Molecular BEC and Atomic+Molecular Superfluids
Deviation from Normal Fermi Gas

\[ T_C = 0.15(1) \varepsilon_F \]

\[ \xi(T=0) \approx 0.41(2) \]

\[ a = \pm \infty \]

\[ \xi = \frac{E}{E_{FG}}, \quad \mu - \text{chemical potential} \]

\[ E_{\text{quasi-particles}}(T) = \frac{3}{5} \varepsilon_F N \frac{5}{2} \sqrt{\frac{2 \pi \Delta^3 T}{\varepsilon_F^4}} \exp\left(-\frac{\Delta}{T}\right) \]

\[ \Delta = \left(\frac{2}{e}\right)^{7/3} \varepsilon_F \exp\left(\frac{\pi}{2 k_F a}\right) \]

\[ E_{\text{phonons}}(T) = \frac{3}{5} \varepsilon_F N \frac{\sqrt{3 \pi^4}}{16 \xi_s^{3/2}} \left(\frac{T}{\varepsilon_F}\right)^4, \quad \xi_s \approx 0.41 \]

Low temperature behaviour of a Fermi gas in the unitary regime

\[ F(T) = \frac{3}{5} \varepsilon_F N \varphi \left( \frac{T}{\varepsilon_F} \right) = E - TS \quad \text{and} \quad \frac{\mu(T)}{\varepsilon_F} \approx \xi_s \approx 0.41(2) \quad \text{for} \quad T < T_C \]

\[ \mu(T) = \frac{dF(T)}{dN} = \varepsilon_F \left[ \varphi \left( \frac{T}{\varepsilon_F} \right) - \frac{2}{5} \frac{T}{\varepsilon_F} \varphi' \left( \frac{T}{\varepsilon_F} \right) \right] \approx \varepsilon_F \xi_s \]

\[ \varphi \left( \frac{T}{\varepsilon_F} \right) = \varphi_0 + \varphi_1 \left( \frac{T}{\varepsilon_F} \right)^{5/2} \]

\[ E(T) = \frac{3}{5} \varepsilon_F N \left[ \xi_s + \xi_s \left( \frac{T}{\varepsilon_F} \right)^n \right] \]

Lattice results disfavor either \( n \geq 3 \) or \( n \leq 2 \) and suggest \( n = 2.5(0.25) \)

This is the same behavior as for a gas of noninteracting (!) bosons below the condensation temperature.
Dilute system of fermionic $^6\text{Li}$ atoms in a harmonic trap

- The number of atoms in the trap: $N = 1.3(0.2) \times 10^5$ atoms divided 50-50 among the lowest two hyperfine states.

- Fermi energy: $\varepsilon_F^{ho} = \hbar \Omega (3N)^{1/3}$; $\Omega = (\omega_x, \omega_y, \omega_z)^{1/3}$

- $\varepsilon_F^{ho}/k_B \approx 1 \mu K$

- Depth of the potential: $U_0 \approx 10\varepsilon_F^{ho}$

- How they measure: energy, entropy and temperature?

\[
P V = \frac{2}{3} E \quad \Rightarrow \quad N\langle U \rangle = \frac{E}{2} \quad \text{- virial theorem}
\]

\[
\n(\r) \quad \text{- local density}
\]

Holds at unitarity and for noninteracting Fermi gas
Comparison with experiment
John Thomas’ group at Duke University,

Entropy as a function of energy (relative to the ground state)
for the unitary Fermi gas in the harmonic trap.

Ratio of the mean square cloud size at B=1200G to
its value at unitarity (B=840G) as a function of
the energy. Experimental data are denoted
by point with error bars.

Theory: Bulgac, Drut, and Magierski
PRL 99, 120401 (2007)

B = 1200G ⇒ 1/k_F a ≈ −0.75
Pairing gap

Spectral weight function: \( A(\tilde{p}, \omega) \)

\[
G^{\text{ret/adv}}(\tilde{p}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega' \frac{A(\tilde{p}, \omega')}{\omega - \omega' \pm i0^+}
\]

\[
G(\tilde{p}, \tau) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega A(\tilde{p}, \omega) \frac{e^{-\omega\tau}}{1 + e^{-\beta\omega}}
\]

From Monte Carlo calcs.

In the limit of independent quasiparticles: \( A(\tilde{p}, \omega) = 2\pi\delta(\omega - E(p)) \)

\( T = 0.1\epsilon_F < T_C \)

\( T = 0.19\epsilon_F > T_C \)

\( (p/p_F)^2 \)

Fermi level

2\(\Delta\)
Results in the vicinity of the unitary limit:
- Critical temperature
- Pairing gap at $T=0$

Note that:
- at unitarity: $\Delta / \varepsilon_F \approx 0.5$
- for atomic nucleus: $\Delta / \varepsilon_F \approx 0.03$

BCS theory predicts:
$$\frac{\Delta(T = 0)}{T_C} \approx 1.7$$
$$\frac{\Delta(T = 0)}{T_C} \approx 3.3$$

This is NOT a BCS superfluid!

Bulgac, Drut, Magierski, PRA78, 023625(2008)
Outside the BCS regime close to the unitary limit, but still before BEC, superconductivity/superfluidity emerge out of a very exotic, non-Fermi liquid normal state.

Monte Carlo calculations

The onset of superconductivity occurs in the presence of fermionic pairs!
Single-particle properties

Effective mass: \( m^* = (1.0 \pm 0.2)m \)

Mean-field potential: \( U = (-0.5 \pm 0.2)\varepsilon_F \)

Weak temperature dependence!

Quasiparticle spectrum extracted from spectral weight function at \( T = 0.1\varepsilon_F \)

Fixed node MC calcs. at \( T=0 \)
**Conclusions**

✓ Fully non-perturbative calculations for a spin $\frac{1}{2}$ many fermion system in the unitary regime at finite temperatures are feasible and apparently the system undergoes a phase transition in the bulk at $T_c = 0.15 \, \epsilon_F$.

✓ Between $T_c$ and $T_0 = 0.23(2) \, \epsilon_F$ the system is **neither superfluid nor follows the normal Fermi gas behavior**. Possibly due to pairing effects.

✓ Results (energy, entropy vs temperature) agree with recent measurements: L. Luo et al., PRL 98, 080402 (2007)

✓ The system at unitarity is **NOT** a BCS superfluid. There is an evidence for the existence of *pseudogap* at unitarity (similarity with high-Tc superconductors).

✓ Description of the system at finite temperatures will pose a challenge for the density functional theory (two temperature scales are present).

✓ Surprisingly at low temperatures the gap extracted from the response function within the independent quasiparticle model accurately reproduce the one obtained from the spectral weight function.
### Superconductivity and superfluidity in Fermi systems

<table>
<thead>
<tr>
<th>System</th>
<th>$T_c$ Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dilute atomic Fermi gases</td>
<td>$10^{-12} – 10^{-9}$ eV</td>
</tr>
<tr>
<td>Liquid $^3$He</td>
<td>$10^{-7}$ eV</td>
</tr>
<tr>
<td>Metals, composite materials</td>
<td>$10^{-3} – 10^{-2}$ eV</td>
</tr>
<tr>
<td>Nuclei, neutron stars</td>
<td>$10^5 – 10^6$ eV</td>
</tr>
<tr>
<td>QCD color superconductivity</td>
<td>$10^7 – 10^8$ eV</td>
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</tbody>
</table>

20 orders of magnitude over a century of (low temperature) physics

Units (1 eV $\approx 10^4$ K)
More details of the calculations:

- Lattice sizes used: $6^3$ – $10^3$.
  Imaginary time steps: $8^3 \times 300$ (high Ts) to $8^3 \times 1800$ (low Ts)

- Effective use of FFT(W) makes all imaginary time propagators diagonal (either in real space or momentum space) and there is no need to store large matrices.

- Update field configurations using the Metropolis importance sampling algorithm.

- Change randomly at a fraction of all space and time sites the signs the auxiliary fields $\sigma(r,\tau)$ so as to maintain a running average of the acceptance rate between 0.4 and 0.6.

- Thermalize for 50,000 – 100,000 MC steps or/and use as a start-up field configuration a $\sigma(x,\tau)$-field configuration from a different $T$.

- At low temperatures use Singular Value Decomposition of the evolution operator $U(\{\sigma\})$ to stabilize the numerics.

- Use 200,000–2,000,000 $\sigma(x,\tau)$- field configurations for calculations.

- MC correlation “time” $\approx 250$ – 300 time steps at $T \approx T_c$. 