MICROSCOPIC CALCULATIONS FOR RARE BETA DECAYS

BY
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Academic Dissertation
for the Degree of
Doctor of Philosophy

To be presented, by permission of the
Faculty of Mathematics and Natural Sciences
of the University of Jyväskylä,
for public examination in Auditorium FYS1 of the
University of Jyväskylä on May 28, 2010
at 12 o’clock noon

Jyväskylä, Finland
May 2010
Preface

The work presented in this thesis was carried out at the Department of Physics of the University of Jyväskylä (JYFL) during the years 2006–2010. I thank Prof. Jouni Suhonen for introducing me to the fascinating field of theoretical nuclear physics and his patient guidance throughout this project. I also wish to express my gratitude to the examiners of this thesis, Dr. Vadim Rodin and Dr. Vladimir I. Tretyak, who found several misprints that have been corrected in the final version of the manuscript.

There are many people that have greatly influenced me during these years in JYFL. There have been several exceptionally inspiring lecturers, such as (but not limited to) Prof. Kari J. Eskola, Dr. Kimmo Kainulainen, Dr. Sami Räsänen, and Prof. Jacek Dobaczewski. There have been countless fellow students who have shared the passion for trying to understand the nature of Nature sparking eye-opening discussions, and balanced the life with their wonderful sense of humor on a daily basis. All these people have my gratitude.

I gratefully acknowledge the financial support by the University of Jyväskylä, the national Graduate School for Particle and Nuclear Physics (GRASPANP), the Ellen and Artturi Nyyssönen Foundation and the Finnish Cultural Foundation. Their funding has not only paid the roof above my head and the food on my plate, but allowed me to participate in and contribute to conferences both in Finland and abroad.

Finally, I wish to thank my mother and my brother and all my friends for their continuing support and encouragement. My late father repeatedly advised me to educate myself as far as I could. Although I think he meant studying medicine or law, I doubt he would be disappointed his son pursued a PhD in nuclear physics.

Jyväskylä, April 2010

Mika Mustonen
Abstract

In this thesis consisting of six publications and an overview part, three cases of rare beta decays are studied using microscopic nuclear models.

Firstly, the half-lives and electron spectra of $^{113}\text{Cd}$ and $^{115}\text{In}$ fourth-forbidden non-unique ground-state-to-ground-state beta decays are studied using two closely related nuclear models: The microscopic quasiparticle-phonon model (MQPM) and the proton-neutron MQPM (pnMQPM), which has been developed as a part of this thesis work. Our results for these rare decays are compared to the available experimental data and are found to agree reasonably well.

As the second application, the partial half-lives of the yet unobserved single-beta decay channels competing with the double beta decay of $^{96}\text{Zr}$ are computed to estimate the possible contamination from these channels to the geochemical double-beta-decay experiments. According to our results obtained by applying the proton-neutron quasiparticle random-phase approximation (pnQRPA), the error stemming from them is still within the experimental uncertainties of the geochemical experiments.

Finally, the recently discovered tiny ultra-low-$Q$-value decay branch of $^{115}\text{In}$ has been investigated in collaboration with the JYFLTRAP group in Jyväskylä and the HADES underground laboratory in Belgium. Our pnMQPM prediction for the half-life is found to differ from the experimentally obtained result by more than an order of magnitude. The various atomic contributions possibly responsible for this discrepancy are discussed.
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The author has derived the formulas for Hamiltonian matrix elements and transition densities in the proton-neutron quasiparticle-phonon model (pnMQPM). The author has written the computer programs for solving the pnMQPM equations, calculating the charge-changing transition densities and evaluating the formulas for highly-forbidden beta decays. The author has carried out the theoretical calculations for publications I, III, IV and VI and written the first draft for publications I, III, V and VI. The author has also contributed to several conference proceedings not included in this thesis.
1 Introduction

Nuclear physics, born a century ago in the famous Rutherford experiment revealing the unexpected existence of the atomic nucleus, still holds many mysteries. The nuclear beta decay is often considered a well-understood part of the nuclear theory. This thesis work concentrates on a few extreme cases of rare beta decays which had not been studied theoretically before.

The single-beta-decay studies are currently not very fashionable, the spotlight shining on the double-beta-decay experiments promising to uncover the true nature of the neutrino – whether it is its own antiparticle or not – as well as its mass. The nuclides studied in the present work are of interest to the experimentalists because of their possible applications in neutrino physics and double-beta-decay studies. $^{115}$In has been envisioned as a real-time detector material for the LENS experiment [1] aiming to measure the Sun’s neutrino luminosity to high precision and hence push forward to expand our knowledge of both our own star and the physics of neutrinos. The COBRA\(^1\) double-beta-decay experiment [2] has used $^{113}$Cd beta decay as one of the test experiments for their room-temperature CdZnTe semiconductor detectors [3]. In $^{96}$Zr, the single beta decay competes with the double beta decay, and is a possible error source for the geochemical measurement of the double-beta-decay half-life.

There exists no practicable nuclear model capable of predicting every observable of every one of the thousands of known nuclides\(^2\). To be able to compute quantities which could be compared to experimental results, a nuclear theorist needs to simplify the nuclear many-body problem by introducing one approximation after another while preserving the essential features of the nucleus. This leads to a myriad of nuclear models, each with distinct capabilities and limitations arising from the chosen approximations.

There are many microscopic models based on different ways of coupling BCS quasiparticles and phonons to form the configuration basis and deriving the Hamiltonian, such as the quasiparticle-phonon nuclear model (QPNM) [5], the microscopic anharmonic vibrator approach (MAVA) [6], its proton-neutron variant (pnMAVA) [7] and the microscopic quasiparticle-phonon model (MQPM) [8]. As a part of this thesis project, the proton-neutron quasiparticle-phonon model (pnMQPM) was developed

\(^1\)Cadmium-zinc-telluride 0-neutrino double-Beta Research Apparatus.

\(^2\)In February 3rd 2010, the total number of nuclides in the NuDat 2 database [4] of Brookhaven National Laboratory was 3175.
as yet another tool to complement this arsenal. Its aim is to better describe the weak-interaction processes such as beta decay and neutrino scattering in odd-mass medium-heavy to heavy spherical (or nearly spherical) nuclei.
2 Describing the nuclear structure

Many difficulties in solving the nuclear many-body problem arise from the inconvenient number of particles. Except for the very lightest nuclei, there are too many particles involved for the problem to be solved by exact methods. Yet there are too few particles in a nucleus for a good statistical description. The interaction between the nucleons is strong and still only approximately known, making the problem even more challenging. The mean-field approximation has been found to be a successful starting point for tackling the nuclear many-body problem.

2.1 Nuclear mean field

In the mean-field approximation nucleons are considered as independent particles moving in the nuclear mean field created by the other nucleons in the nucleus. More formally, to solve the many-body Hamiltonian $H = T + V$, where $T$ stands for the kinetic energy and $V$ for the potential energy terms, a mean-field potential $V_{\text{m.f.}}$ is chosen so that both the Hamiltonian $H_{\text{m.f.}} = T + V_{\text{m.f.}}$ is solvable and the residual interaction $V_{\text{res}} = V - V_{\text{m.f.}}$ is sufficiently small that one can apply the perturbation theory. Essentially this approach converts the problem of strongly interacting particles to the easier problem of weakly interacting mean-field quasiparticles.

The mean field potential can be generated e.g. by the iterative Hartree-Fock procedure. Alternatively, an effective potential can be taken as the mean field: for example the harmonic oscillator potential or, as in this work, the more realistic phenomenological Woods-Saxon potential (Figure 2.1)

$$v_{\text{WS}}(r) = -\frac{V_0}{1 + e^{(r-R)/a}},$$

(2.1)

where we have adopted the parametrization of Ref. [9]: $R = 1.27 \text{ fm} \times A^{1/3}$ for the nuclear radius, $a = 0.67 \text{ fm}$ for the surface diffuseness and

$$V_0 = \left( 51 \pm 33 \frac{N-Z}{A} \right) \text{ MeV},$$

(2.2)

where the plus sign is selected for protons and the minus sign for neutrons, for the depth of the potential. The Woods-Saxon potential is known to successfully reproduce
the observed shell structure in the nuclei, when the spin-orbit interaction is taken into account. For protons the Coulomb interaction must also be included, but it does not modify the shell structure as dramatically.

In the calculations of this work, some energies of the mean-field states near the Fermi surface have been adjusted by hand so that the BCS quasiparticle spectrum better agrees with the experimental energy level data. This is justified since the Woods-Saxon potential is a global parametrization for the mean-field. Special care has been taken so that no unphysical adjustments, such as changing the order of spin-orbit partners\(^1\), are made.

The problem is further simplified by including only a limited set of single-particle states in the valence space. In this work, the valence space is selected so that it contains two to three major shells (see Chapter 4) around the Fermi levels for protons and neutrons. The fully occupied states below the valence space in energy are approximated as an inert core that does not directly interact with the valence space particles.

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\(^1\)Spin-orbit partners differ in total angular momentum by one unit but share the same principal quantum number \(n\) and the same orbital angular momentum quantum number \(l\), for example \(1d_{3/2}\) and \(1d_{5/2}\).
2.2 Bardeen-Cooper-Schrieffer theory for nuclei

There is plenty of experimental evidence of the pairing phenomenon in nuclei, i.e. the tendency of the nuclear Hamiltonian to favour pairs of nucleons coupled to zero total angular momentum. Perhaps the most striking example of such evidence is the fact that the ground-state of every known even-even nucleus has the angular momentum and parity $0^+$. Other evidence include e.g. the energy gap between the ground state and the lowest excitations present only in the even-even nuclei and the odd-even effect. The analogy to the Cooper pairs in the Bardeen-Cooper-Schrieffer (BCS) theory for superconducting metals [10] was quickly recognized in the late 1950's by Bohr, Mottelson and Pines [11], and the BCS approach adapted to nuclei soon became one of the standard tools of microscopic nuclear theory.

The BCS quasiparticle creation and annihilation operators are formed as linear combinations of the mean-field quasiparticle creation and annihilation operators

\[
\begin{align*}
\hat{a}_{\beta}^\dagger &= u_{\beta} c_{\beta}^\dagger + v_{\beta} \tilde{c}_{\beta}^\dagger, \\
\hat{a}_{\beta} &= u_{\beta} \tilde{c}_{\beta} - v_{\beta} c_{\beta}^\dagger.
\end{align*}
\]

(2.3)

This is known as the Bogolyubov-Valatin transformation after the two physicists who first introduced it in [12, 13]. Here the notation of Baranger [14] is adopted: A Roman index stands for the set of the principal quantum number $n$, the orbital angular momentum $l$ and the total angular momentum $j$, e.g. $b = \{n_b, l_b, j_b\}$. The corresponding Greek index also includes the projection quantum number $m$, so that e.g. $\beta = \{n_b, l_b, j_b, m_\beta\}$. The operator $c_{\beta}^\dagger$ is the particle creation operator and $\tilde{c}_{\beta}$ is the corresponding time-reversed annihilation operator. The occupation amplitude $v_b$ and the unoccupation amplitude $u_b$ are to be solved via a variational procedure where the ground-state energy is minimized (See e.g. [15] for details).

The BCS ground state, formally written as the ansatz

\[
|\text{BCS}\rangle = \prod_{\alpha > 0} (u_{\alpha} - v_{\alpha} c_{\alpha}^\dagger \tilde{c}_{\alpha}^\dagger) |\text{CORE}\rangle,
\]

(2.4)

acts as the vacuum for the quasiparticles. The index $\alpha$ runs over all the like-nucleon states having a positive projection quantum number $m_\alpha$. The $|\text{CORE}\rangle$ represents the inert nuclear core, i.e. the filled single-particle states energetically below the selected model space. The particle number is not a good quantum number for the BCS ground state. The condition that the average particle number equals to the number of valence

\footnote{From this point on in this thesis the BCS quasiparticles are referred to as \textit{quasiparticles} and the mean-field quasiparticles simply as \textit{particles}.}
nucleons in the reference nucleus, ground state of which the BCS vacuum is aimed to describe, is used as a constraint in the variational procedure.

The nuclear Hamiltonian with a two-particle interaction is

\[ H = \sum_\alpha \varepsilon_\alpha c_\alpha^\dagger c_\alpha + \frac{1}{4} \sum_{\alpha\beta\gamma\delta} \bar{v}_{\alpha\beta\gamma\delta} c_\alpha^\dagger c_\beta^\dagger c_\delta c_\gamma, \]

(2.5)

where the antisymmetrized two-body matrix elements \( \bar{v}_{\alpha\beta\gamma\delta} \) are obtained in this work from the Bonn one-boson-exchange potential using the \( G \)-matrix technique [16]. When the Hamiltonian is transformed to the quasiparticle picture using the Bogolyubov-Valatin transformation, it takes the form

\[ H = H_{11} + H_{20} + H_{02} + H_{22} + H_{40} + H_{04} + H_{31} + H_{13}, \]

(2.6)

where each term \( H_{mn} \) is proportional to a normal-ordered product of \( m \) quasiparticle creation and \( n \) quasiparticle annihilation operators. The ground-state energy term has been omitted since only the relative energies of the states are of interest in this work. The minimization of the ground-state energy in the BCS procedure causes the terms \( H_{20} \) and \( H_{02} \) to vanish. Physically this means that there are no quasiparticle pair excitations across the Fermi surface. This indicates that the transformation absorbs a large part of the residual interaction \( V_{\text{res}} \) into the very structure of the quasiparticles.

In the BCS calculations of this work the interaction matrix elements were scaled by a constant \( g_{\text{pair}} \) for protons and neutrons separately so that the lowest quasiparticle energy matched the experimental pairing gap, which was calculated from the experimental separation energies of [17, 18, 19, 20, 21] using the linear approximation formulas of [22]. These adjustments were small in all our applications, i.e. the scaling constants \( g_{\text{pair}} \) were close to unity.

### 2.3 Proton-neutron quasiparticle random-phase approximation

The proton-neutron quasiparticle random-phase approximation (pnQRPA) [23] describes the states of the adjacent isobars of the BCS reference nucleus as proton-neutron-quasiparticle-pair excitations. These excitations, called pnQRPA phonons, are created by the operator

\[ Q_\omega^\dagger = \sum_{pn} \left( X_{pn}^\omega [a_p^\dagger a_n^\dagger]_{J_\omega M_\omega} + Y_{pn}^\omega [\hat{a}_p \hat{a}_n]_{J_\omega M_\omega} \right), \]

(2.7)

where the index \( \omega \) contains the angular momentum \( J_\omega \), the projection quantum number \( M_\omega \) and the parity \( \pi_\omega \) of the phonon, and the additional index \( k_\omega \) enumerating
the different states with the same \( J_\omega, M_\omega \) and \( \pi_\omega \). The forward-going amplitude \( X^\omega_{pn} \) and the backward-going amplitude \( Y^\omega_{pn} \) are to be determined by diagonalizing the pn-QRPA Hamiltonian. In the quasi-boson approximation the pnQRP phonon creation and annihilation operators satisfy the commutation relations

\[
\left[ Q_\omega, Q^\dagger_{\omega'} \right] \approx \delta_{\omega\omega'} \quad \text{and} \quad \left[ Q_\omega, Q_{\omega'} \right] = 0 = \left[ Q^\dagger_{\omega}, Q^\dagger_{\omega'} \right]
\] (2.8)

and act therefore to a good approximation as bosons.

The pnQRP ground state is defined as the vacuum for the pnQRP phonons, i.e.

\[
Q_\omega |\text{pnQRP}\rangle = 0
\] (2.9)

The pnQRP ground state is approximately equal to the BCS ground state as long as the backward-going amplitudes remain significantly smaller than the forward-going ones. If the requirement \( |Y^\omega_{pn}| \ll |X^\omega_{pn}| \) is not met, the RPA description breaks down.

The pnQRP equations can be derived using the equations-of-motion method [24], and they can be written in a convenient matrix form

\[
\begin{pmatrix}
A & B \\
-B^* & -A^*
\end{pmatrix}
\begin{pmatrix}
X^\omega \\
Y^\omega
\end{pmatrix}
= E_\omega
\begin{pmatrix}
X^\omega \\
Y^\omega
\end{pmatrix}
\] (2.10)

A useful procedure for solving eigenvalue problems of this type, developed by Ullah and Rowe [25], is applied in our computational codes.

In our applications to \(^{113}\text{Cd}\) and \(^{115}\text{In}\), the pnQRP spectra were fine-tuned to better correspond to the experimentally observed energy spectra by scaling the interaction matrix elements by the constants \( g_{pp} \) (particle-particle matrix elements) and \( g_{ph} \) (particle-hole matrix elements). However, the final results of the beta decay calculations were not sensitive to these adjustments in any of our applications. The corresponding procedure was used for scaling the QRP matrix elements in the MQPM calculation [8].

The pnQRP is known to satisfy the Ikeda sum rule [26]. This fact motivated us to develop a variation of the MQPM using the pnQRP phonons instead of the QRP phonons hoping that it might improve the description of the weak-interaction transitions, such as beta decays and neutrino-nucleus scattering reactions.

2.4 Proton-neutron microscopic quasiparticle-phonon model

In the microscopic quasiparticle-phonon model (MQPM), first presented in [27] and further refined in [8], the basis of the states of a fixed total angular momentum and
parity $j^\pi$ consists of the one-quasiparticle states and three-quasiparticle states constructed by coupling one-quasiparticle states and QRPA phonons to $j^\pi$. The proton-neutron variant of the MQPM (pnMQPM) takes the same approach but using the pnQRPA phonons. The pnMQPM excitation operator reads

$$\Gamma_i(jm) = \sum_p C^i_p a^\dagger_{p,m} + \sum_{\omega} D^{i}_{\omega} [a^\dagger_{n,\omega} Q^\dagger_{\omega}]_{jm}$$

or

$$\Gamma_i(jm) = \sum_n C^{i}_{n} a^\dagger_{n,m} + \sum_{\omega} D^{i}_{\omega} [a^\dagger_{p,\omega} Q^\dagger_{\omega}]_{jm}$$

in the case of the proton-odd or the neutron-odd nucleus, respectively. Naturally, the first summation runs over only those quasiparticle states that have the angular momentum and parity $j^\pi$, and the second one over the combinations of quasiparticles and phonons that can be coupled to $j^\pi$. The amplitudes $C^i_a$ and $D^{i}_{a\omega}$ are to be determined by diagonalizing the Hamiltonian.

In practice, the basis is truncated by taking into account only the lowest-energy phonons. This is physically justified, since the high-energy states are expected to interfere very little with the low-lying states. A suitable cut-off value for the phonon energy was found by observing the convergence of the amplitudes of the dominant components.

Figure 2.2: Squared amplitudes of the four largest components of the $^{135}$I lowest $11/2^+$ state pnMQPM wave function as functions of the cut-off energy for the pnQRPA phonons. The convergence at 4 MeV corresponds to 51 pnQRPA phonons.
The one-quasiparticle states are orthogonal to each other and to the three-quasiparticle states. However, the three-quasiparticle states are not orthogonal to each other and form an overcomplete set. The overlap between two three-quasiparticle states is

$$\langle |[a_n^+Q_{\omega'}^+]_{jm}| \rangle = \delta_{nn'}\delta_{\omega\omega'} + K^{(p)}(n\omega'n'\omega';j)$$ (2.13)

in the case of the neutron-odd nucleus or

$$\langle |[a_p^+Q_{\omega'}^+]_{jm}| \rangle = \delta_{pp'}\delta_{\omega\omega'} + (-1)^{j_p+j_p'+j_\omega+j_\omega'}K^{(n)}(p\omega'p';j)$$ (2.14)

in that of the proton-odd nucleus. The auxiliary expressions $K^{(p)}(n_1\omega_1n_2\omega_2;j)$ and $K^{(n)}(p_1\omega_1p_2\omega_2;j)$ are defined as

$$K^{(p)}(n_1\omega_1n_2\omega_2;j) = \delta_{pp'}\delta_{\omega\omega'} + \frac{1}{j_2} \sum_p \left( \begin{array}{ccc} j & j_1 & j_2 \\ j_p & j_n & j_\omega \\ n_1 & n_2 & \omega_1 \\ \omega_2 & \omega_2 & n_1 \end{array} \right) \left( X_{p_2n_2}^{\omega_2}X_{p_1n_1}^{\omega_1} - \delta_{jj_2}X_{p_2n_1}^{\omega_2}Y_{p_1n_2}^{\omega_1} \right)$$ (2.15)

and

$$K^{(n)}(p_1\omega_1p_2\omega_2;j) = \delta_{pp'}\delta_{\omega\omega'} + \frac{1}{j_2} \sum_n \left( \begin{array}{ccc} j & j_1 & j_2 \\ j_n & j_p & j_\omega \\ \omega_1 & \omega_1 & p_1n_1 \end{array} \right) \left( X_{p_1n_1}^{\omega_1}X_{p_2n_2}^{\omega_2} - \delta_{jj_2}Y_{p_1n_2}^{\omega_2}X_{p_1n_1}^{\omega_1} \right)$$ (2.16)

The equations-of-motion method [24] leads to the generalized eigenvalue equation

$$H\psi_k = E_kN\psi_k,$$ (2.17)

where the matrix $N$ is the overlap matrix, $H$ is the Hamiltonian matrix and $\psi_k$ is the eigenvector corresponding to the energy eigenvalue $E_k$. The components of $\psi_k$ are the coefficients $C^k_p$ and $D^k_{n\omega}$ (or $C^k_n$ and $D^k_{p\omega}$) of Eqs. (2.11) and (2.12).

The Hamiltonian matrix elements in the pnMQPM basis are listed in Appendix A. The MQPM and the pnMQPM share the important feature that they both start from the Hamiltonian of Eq. (2.6) with simple interpretation of its terms: Interaction between two phonons ($H_{22}$), between one-quasiparticle and three-quasiparticle configurations ($H_{31} + H_{13}$) etc. All these terms are treated on equal footing, differentiating the MQPM and the pnMQPM from many other quasiparticle-phonon approaches.

The differences between the MQPM and pnMQPM arise from two factors: Firstly, the three-quasiparticle part of the configuration space is different between these two
models. For a proton-odd nucleus, the MQPM three-quasiparticle configuration space contains the three-proton configurations and one-proton-two-neutron configurations. In the pnMQPM only one-proton-two-neutron configurations are included due to the proton-neutron structure of the pnQRPA excitation. This is the primary weakness of the pnMQPM model, and for the states where three-proton contributions are important, the pnMQPM description is inevitably inaccurate.

The second difference is that in the RPA step the adjustments are made to a different set of experimental data. This can be an advantage to the pnMQPM, when there are problems describing the reference nucleus spectrum using the QRPA. One example of such a case is $^{135}\text{I}$, where the MQPM fails to produce the correct energy spectrum due to the too high QRPA energy of $^{134}\text{Te} 2^+_1$ state [28]. This nucleus was used as one of our early test cases for the pnMQPM; the results for the lowest states are presented in Figure 2.3.

Our adopted method [30] for dealing with the overcompleteness of the pnMQPM basis is the same as the one used with the MQPM. First the overlap matrix $N$ is diagonalized to find a new, orthogonal basis:

$$N u^{(k)} = n_k u^{(k)},$$  \hspace{1cm} (2.18)
where the index $k$ enumerates the different eigenstates. Ideally, the states to be discarded as spurious would have a zero eigenvalue. However, due to the fact that not all the pnQRPA phonons are used in building the basis, and also (less importantly) to the fact that there is likely to be some numerical inaccuracy, these states often have a small non-zero eigenvalue. In practice, one needs to set a suitable upper limit for the eigenvalues which are to be discarded. The physical validity of the choice for this cut-off value was monitored in our application by substitution of the computed wave functions back in the original generalized eigenvalue equation.

The new complete set of orthonormal basis states are

$$|\tilde{k}\rangle = \frac{1}{\sqrt{n_k}} \sum_i u_i^{(k)} |i\rangle,$$

(2.19)

where the tilde emphasizes that the vector belongs to the set of new basis vectors, and the index $i$ runs over the basis states of the overcomplete basis. In the new basis the original generalized eigenvalue equation transforms into an ordinary eigenproblem

$$\tilde{H} g^{(v)} = \lambda_v g^{(v)},$$

(2.20)

where the Hamiltonian matrix elements in the new basis are

$$\tilde{H}_{ab} = \frac{1}{\sqrt{n_a n_b}} \sum_{ij} (u^{(a)}_i)^* u^{(b)}_j \langle i | H | j \rangle.$$  

(2.21)

The eigenvalues $\lambda_v$ are now the energies of the (pn)MQPM states. The (unnormalized) coefficients of the (pn)MQPM wave functions in the original basis can be identified from the equation

$$|v\rangle = \sum_i \sum_a g^{(v)}_a \frac{1}{\sqrt{n_a}} u^{(a)}_i |i\rangle,$$

(2.22)

where again the index $i$ runs over the basis vectors of the overcomplete basis and $a$ runs over the dimension of the restricted complete basis.
Describing the nuclear structure
3 Nuclear beta decay

The general formalism of the beta-decay theory is developed in detail in the comprehensive treatment by Behrens and Bühring [31]. Less detailed introductions to calculating the allowed and unique forbidden decays can be found in many text books, e.g. in [15]. In this chapter the general beta-decay theory of [31] is superficially introduced and the results of [31] are connected to the nuclear models used in this work.

The famous first theoretical work to quantitatively explain the beta decay was the theory by Fermi [32]. It was inspired by the quantum-mechanical treatment of electromagnetic radiation, and consequently the form of a vector interaction was assumed for the decay mechanism. Pauli's neutrino hypothesis was a key ingredient in the theory, and its success lead to the general acceptance of the elusive particle's existence long before it was experimentally observed. Fermi's theory was soon extended by Gamow and Teller [33] by including the possibility of emitting the electron-neutrino pair in a spin triplet state. After the discovery of the parity violation in the beta-decay experiment by Wu et al [34], the phenomenological theory for beta decay reached its final vector-minus-axial-vector ("V − A") form.

Unlike the quantum electrodynamics, the V − A theory is not renormalizable. This problem was solved by the gauge theory unifying the weak and electromagnetic interaction introduced by Glashow, Weinberg and Salam (See e.g. [35]). Predictions of this electroweak theory have been verified by countless experiments, and the one experimentally yet unobserved particle predicted by the theory, the famous Higgs boson, is expected to reveal itself in the Large Hadron Collider (LHC) experiment at CERN.

3.1 Phenomenological V − A theory

In the standard model of electroweak interactions, the beta decay is a semi-leptonic process mediated by a charged gauge boson W− or W+ (See Figure 3.1). Due to the high mass of the gauge bosons (roughly 80 GeV), the weak interaction has an extremely short range; it is point-like even on the nuclear scale. From the nuclear-theory point of view it is a very good approximation to consider it as an effective four-point interaction. This phenomenological vector-minus-axial-vector ("V − A") interaction of the hadronic and the leptonic current has proven to be a tractable approach at the low energies of the nuclear beta decay.
For the leptonic current the vector coupling and the axial vector coupling are of the same strength, but due to the internal structure of the hadrons and the renormalization effects of the strong interaction, the axial-vector coupling is modified in the case of the hadronic current. The renormalization of the hadronic axial-vector current is, however, small in the sense that the value of the axial-vector coupling constant $g_A$ is not far from unity. In this work, the value $g_A = 1.25$ is adopted. The fact that the axial current is only slightly renormalized is known in the standard model as the partially conserved axial-vector current (PCAC) hypothesis.

The vector part of the current is nevertheless unmodified by the presence of the strong interaction. It is believed to be “protected” by the electromagnetic interaction [36]. This is known as the conserved vector current (CVC) hypothesis, and it asserts that the value of the vector coupling constant is $g_V = 1$.

More formally, the Hamiltonian density in the $V − A$ theory of beta decay is

$$H_\beta(x) = -\frac{G_F}{\sqrt{2}} \left[ \bar{\psi}_p(x) \gamma_\mu (1 + \lambda \gamma_5) \psi_n(x) \bar{\psi}_e(x) \gamma^\mu (1 - \gamma_5) \psi_\nu(x) + \text{H.c.} \right]$$  \hspace{1cm} (3.1)

where $J_\mu(x)$ is the hadron current (the nuclear current) and $L_\mu(x)$ is the lepton current operator. The constant $G_F$ is the Fermi coupling constant. The constant $\lambda = -g_A/g_V$ in the hadron current operator is the ratio between the axial-vector and vector coupling constants discussed above.
3.2 General formalism for $\beta^-$ decays

The probability for emitting an electron with the total energy in the interval $[W_e, W_e + dW_e]$ is

$$P(W_e)dW_e = \frac{G_F^2}{(\hbar c)^6} \frac{1}{2\pi^4 \hbar} F_0(Z, W_e) C(W_e) p_e c W_e (W_0 - W_e)^2 dW_e, \quad (3.2)$$

where $Z$ is the charge of the daughter nucleus, $p_e = \sqrt{W_e^2 - (m_e c^2)^2}$ is the electron momentum and $W_0$ is the maximum total energy of the electron. The Fermi function $F_0(Z, W_e)$ approximates the effect of the Coulomb interaction between the nucleus and the emitted electron on the beta spectrum. It is defined as the ratio of the absolute squares of the relativistic Coulomb wave function and the free lepton wave function at the nuclear radius [15]. Introducing modifications to the Fermi function is the natural way of taking atomic corrections into account, for example by the Rose prescription [37] where the electron screening is taken into account by replacing the Fermi function with

$$F_0^{\text{screened}}(Z, W_e) \approx \frac{\hat{p}_e \hat{W}_e}{p_e W_e} F_0(Z, W_e), \quad (3.3)$$

where $\hat{W}_e = W_e - V_0$ and $\hat{p}_e = \sqrt{\hat{W}_e^2 - (m_e c^2)^2}$. Here $V_0$ is the difference between the Coulomb potential energy and the exact potential energy of the continuum electron at the nuclear radius, and it has been studied in more detail in [38]. The shape factor $C(W_e)$ contains the details of the nuclear structure. It is a constant for allowed decays, and in the case of forbidden decays it modifies the beta spectrum shape in addition to the total decay probability. The kinematical factor $p_e c W_e (W_0 - W_e)^2 dW_e$ arises from the available final-state-lepton phase space with the approximation of zero neutrino mass.

Integrating the probability density (3.2) over all possible electron energies yields the decay rate, and the half-life is then simply

$$T_{1/2} = \kappa \left( (m_e c^2)^{-5} \int_{m_e c^2}^{W_0} F_0(Z, W_e) C(W_e) p_e c W_e (W_0 - W_e)^2 dW_e \right)^{-1}. \quad (3.4)$$

Formally the constant $\kappa$ is

$$\kappa = \frac{2\pi^3 \hbar \ln 2}{(m_e c^2)^5 G_F^2 / (\hbar c)^6}, \quad (3.5)$$

but in practice it is slightly renormalized by the so-called inner radiative corrections. Therefore it is more accurate to use the value $\kappa = 6147 \text{s}$ obtained from an extensive survey of superallowed $\beta$ decays [39].
To evaluate the shape factor, the first step in the treatment of [31] is to make a
multipole expansion of the nuclear current

\[ (-i)\langle f|V_\mu(0) + A_\mu(0)|i\rangle \gamma_0 \gamma^\mu = \sum_{KLS} (-1)^{J_f-M_f+M} (-i)^L \sqrt{4\pi} \hat{J}_i \]
\[ \times \left( \begin{array}{ccc} J_f & K & J_i \\ -M_f & M & M_i \end{array} \right) T_{KLS}^{-M}(\hat{q}) \frac{(qR/\hbar)^L}{(2L+1)!!} F_{KLS}(q^2), \quad (3.6) \]

where \( q \) is the momentum transfer, \( R \) is the nuclear radius, \( T_{KLS}^{-M}(\hat{q}) \) is an operator
acting on the lepton spinors and \( F_{KLS}(q^2) \) are the nuclear form factors containing all
the details of the nuclear structure. The form factors are then expanded as a power
series

\[ F_{KLS}(q^2) = \sum_n \frac{(-1)^n (2L + 1)!!}{(2n)!!(2L + 2n + 1)!!} \left( \frac{qR}{\hbar} \right)^{2n} F_{KLS}^{(n)}. \quad (3.7) \]

Due to the huge mass of the nucleus compared to the electron, the momentum transfer \( q \) is so small that a form factor can be approximated with its leading term \(^1\)
\( F_{KLS}^{(0)} = F_{KLS}(q = 0) \). Taking the non-zero nuclear charge into account also introduces additional form factors \( F_{KLS}(k_e, 1, 1, 1) \), which are sensitive to the electron relativistic quantum number \( k_e \) and the nuclear charge distribution. In this work, a uniform spherical charge distribution is assumed for the nucleus.

The form factors enter the formulas for observables only as certain highly complicated linear combinations \( M_K(k_e, k_\nu) \) and \( m_K(k_e, k_\nu) \). The beta-decay shape factor has the form

\[ C(W_e) = \sum_{k_e k_\nu K} \frac{F_{k_e-1}(Z, W_e)}{F_0(Z, W_e)} \left[ \frac{M_K^2(k_e, k_\nu) + m_K^2(k_e, k_\nu)}{M_K(k_e, k_\nu) m_K(k_e, k_\nu)} \right], \quad (3.8) \]

where \( K \) is the transferred angular momentum and \( k_e \) and \( k_\nu \) are the absolute values
of the relativistic electron and neutrino quantum numbers \( k_e \) and \( k_\nu \). The Coulomb function \( \mu_{k_e} \) can be approximated as unity in the case of the nuclear beta decay and \( \gamma_{k_e} = \sqrt{k_e^2 - (\alpha Z)^2} \), where \( \alpha \) is the fine-structure constant. \( F_{k_e-1}(Z, W_e) \) is the
generalized Fermi function, for which we use the definition

\[ F_{k_e-1}(Z, W_e) = 4^{k-1} (2k)(k+\gamma_k)[(2k-1)!!]^2 e^{\gamma_k y} \left( \frac{2p_e R}{\hbar} \right)^{2(\gamma_k - k)} \left( \frac{1}{\Gamma(1 + 2\gamma_k)} \right)^2, \quad (3.9) \]

where \( y = \alpha Z W_e/(p_e c) \) and \( \Gamma(z) \) is the usual gamma function. This definition is equal
to \( (k + \gamma_k) F_{k-1}/(2k) \) in the notation of [31].

\(^1\)In [31] these are called "form factor coefficients".
3.2 General formalism for $\beta^-$ decays

<table>
<thead>
<tr>
<th>$\Delta J$</th>
<th>$\pi_i \pi f$</th>
<th>name</th>
<th>relevant form factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 or 1</td>
<td>+1</td>
<td>allowed</td>
<td>$V F_{10}^{(0)}, A F_{011}^{(0)}$</td>
</tr>
<tr>
<td>0 or 1</td>
<td>−1</td>
<td>1st forbidden non-unique</td>
<td>$V F_{10}^{(0)}, A F_{111}^{(0)}, A F_{211}^{(0)}$, $A F_{011}^{(0)} V F_{10}^{(0)}(k_e, 1, 1, 1), A F_{111}^{(0)}(k_e, 1, 1, 1)$</td>
</tr>
<tr>
<td>2</td>
<td>−1</td>
<td>1st forbidden unique</td>
<td>$A F_{211}^{(0)}$</td>
</tr>
<tr>
<td>2</td>
<td>+1</td>
<td>2nd forbidden non-unique</td>
<td>$V F_{211}^{(0)}, V F_{221}^{(0)}, A F_{221}^{(0)}, A F_{321}^{(0)}$, $V F_{221}^{(0)}(k_e, 1, 1, 1), A F_{221}^{(0)}(k_e, 1, 1, 1)$</td>
</tr>
<tr>
<td>3</td>
<td>+1</td>
<td>2nd forbidden unique</td>
<td>$A F_{321}^{(0)}$</td>
</tr>
<tr>
<td>3</td>
<td>−1</td>
<td>3rd forbidden non-unique</td>
<td>$V F_{321}^{(0)}, V F_{331}^{(0)}, A F_{331}^{(0)}, A F_{431}^{(0)}$, $V F_{331}^{(0)}(k_e, 1, 1, 1), A F_{331}^{(0)}(k_e, 1, 1, 1)$</td>
</tr>
<tr>
<td>4</td>
<td>−1</td>
<td>3rd forbidden unique</td>
<td>$A F_{431}^{(0)}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$K$</td>
<td>$(-1)^K$</td>
<td>$K$th forbidden non-unique</td>
<td>$V F_{K,K-1,1}^{(0)}, V F_{K,K0}^{(0)}, A F_{KK1}^{(0)}, A F_{K+1,K1}^{(0)}$, $V F_{K,K1}^{(0)}(k_e, 1, 1, 1), A F_{K+1,K1}^{(0)}(k_e, 1, 1, 1)$</td>
</tr>
<tr>
<td>$K+1$</td>
<td>$(-1)^K$</td>
<td>$K$th forbidden unique</td>
<td>$A F_{K+1,K1}^{(0)}$</td>
</tr>
</tbody>
</table>

Table 3.1: Classification scheme for beta decay.

The order-of-magnitude considerations of the form factors, discussed in [31], imply that the most important form factors are those presented in Table 3.1. The other form factors are suppressed by additional powers of $WR/hc, pR/h$ or $\alpha Zm_e cR/h$, or they are linked to the small components of the relativistic wave function. This yields the tabulated classification scheme for beta decays: Knowing only the difference in the angular momentum and parity of the initial and final state one can see which form factors are needed and, furthermore, to which range the log $ft$ value of the decay roughly falls.

The form factors $V/A F_{KLS}^{(0)}$ can be associated with the nuclear matrix elements as described in [31] using the *impulse approximation*. This means that the decaying nucleon is assumed to behave like a free nucleon at the moment of the decay with all the other nucleons acting only as spectators. Then the nuclear matrix elements can be related to the form factors simply as

$$R^L V F_{KLS}^{(0)} = (-1)^{K-L} g_V V M_{KLS}^{(0)}$$  \hspace{1cm} (3.10)

and

$$R^L A F_{KLS}^{(0)} = (-1)^{K-L+1} g_A A M_{KLS}^{(0)}.$$  \hspace{1cm} (3.11)

An important additional simplification occurs in the case that the angular momentum change is $K+1$ and the product of initial and final state parities is $(-1)^K$: Then there
is only one relevant form factor to consider, and in fact all but one of the relevant nuclear matrix elements, \( M = A^4 \mathcal{M}^{(0)}_{K+1,K,1} \), are trivially zero, and the formalism is dramatically simplified. Such decays are called *unique* decays, and for them the half-life can be expressed in the simple form

\[
T_{1/2} = \frac{1}{M^2 f_K(W_0, Z, R)},
\]

(3.12)

where the factor

\[
f_K(W_0, Z, R) = \frac{g_3^2}{\kappa(hc)^2 (m_e c)^5} (2K)!! \int_{m_e c^2}^{W_0} dW_e p_e c W_e \times \sum_{k_e + k_u = K+2} F_{k_e-1}(Z, W_e) \frac{[W_e^2 - (m_e c)^2]^{k_e-1} (W_0 - W_e)^{2k_u}}{(2k_e - 1)!(2k_u - 1)!}
\]

(3.13)

has no other dependence on the nucleus than the \( Q \) value, the charge and the nuclear radius.

In this work, the formalism was used in a streamlined form published in [40]. In the included publications the following shorthand notation for the nuclear matrix elements relevant to the non-unique transitions was adopted:

\[
M_1 = V^4 \mathcal{M}^{(0)}_{K,K-1,1},
\]

(3.14a)

\[
M_2 = V^4 \mathcal{M}^{(0)}_{KK0},
\]

(3.14b)

\[
M_2^{(k_e)} = V^4 \mathcal{M}^{(0)}_{KK0}(k_e, 1, 1, 1),
\]

(3.14c)

\[
M_3 = A^4 \mathcal{M}^{(0)}_{KK1},
\]

(3.14d)

\[
M_3^{(k_e)} = A^4 \mathcal{M}^{(0)}_{KK1}(k_e, 1, 1, 1),
\]

(3.14e)

\[
M_4 = A^4 \mathcal{M}^{(0)}_{K+1,K,1},
\]

(3.14f)

\[
M_5 = A^4 \mathcal{M}^{(0)}_{000},
\]

(3.14g)

\[
M_6 = A^4 \mathcal{M}^{(0)}_{011} \text{ and}
\]

(3.14h)

\[
M_6^{(k_e)} = A^4 \mathcal{M}^{(0)}_{011}(k_e, 1, 1, 1).
\]

(3.14i)

### 3.3 Nuclear matrix elements in the adopted models

To relate the nuclear matrix elements \( V^4 A^4 \mathcal{M}^{(0)}_{KLS} \) to the nuclear wave functions, it is practical to decompose them (for \( \beta^- \) decay) as

\[
V^4 A^4 \mathcal{M}^{(0)}_{KLS} = \frac{\sqrt{4\pi}}{J_1} \sum_{pn} V^4 A^4 m_{KLS}(pn) \langle \psi_f || [c_{F}^T c_n]_K || \psi_i \rangle,
\]

(3.15)
3.3 Nuclear matrix elements in the adopted models

where the reduced single-particle matrix elements $V_{m_{KLS}(pn)}$ are

$$V_{m_{KLS}(pn)} = \hat{K}^{-1}(\bar{p}||T_{KLS}||n) \quad (3.16)$$

and

$$A_{m_{KLS}(pn)} = \hat{K}^{-1}(\bar{p}||\gamma_5 T_{KLS}||n). \quad (3.17)$$

The spherical components of the operator $T_{KLS}$ are

$$T_{KLSM} = \begin{cases} i^L r^L Y_{LM} \delta_{KL}, & S = 0 \\ i^L (-1)^{L+1-K} r^L [Y_L \bar{\sigma}]_{KM}, & S = 1. \end{cases} \quad (3.18)$$

The reduced single-particle matrix elements can be evaluated by taking the large components $G_{nljm}(\vec{r})$ of the single-particle wave functions

$$\phi_{nljm}(\vec{r}) = \frac{G_{nljm}(\vec{r})}{F_{nljm}(\vec{r})} \quad (3.19)$$

to be solutions of the non-relativistic Schrödinger equation for the harmonic oscillator. The details of this procedure and the resulting rather complex formulas can be found in [40].

The charge-changing transition densities (CCTDs) $(\psi_f || [c_p^\dagger \tilde{c}_n]_K || \psi_i)$ depend on the applied nuclear model. For the pnQRPA, explicit expressions can be found in [15], and for the MQPM in [8]. The formulas for evaluating the CCTDs for the pnMQPM wave functions, derived as part of this thesis work, are presented in Appendix B.
Nuclear beta decay
4 Applications to rare beta decays

Most of the observed beta decays are allowed, first-forbidden or second-forbidden decays with MeV-scale $Q$ values. In this work we have studied a few selected cases of rare beta decays with an exceptionally high degree of forbiddenness or an ultra-low $Q$ value.

The highly-forbidden non-unique beta decays have been observed in only a few isotopes: $^{113}$Cd decays exclusively via a fourth-forbidden non-unique beta minus decay. $^{115}$In has a recently-discovered tiny alternative decay channel with a record-low $Q$ value discussed in detail later in this chapter. For $^{50}$V only fourth-forbidden non-unique electron capture and $\beta^-$ decays to excited states in the neighboring isobars have been observed, but the sixth-forbidden non-unique decays directly to their ground states are energetically possible. In $^{48}$Ca and $^{96}$Zr highly-forbidden beta decays compete with double beta decay. The case of $^{48}$Ca has been investigated with the nuclear shell model in [41] and [42], but the beta decays of $^{113}$Cd, $^{115}$In and $^{96}$Zr were theoretically unexplored before the work presented in this thesis.

The decays with low $Q$ values have raised interest because of their possible use in detecting the neutrino mass by observing the shape of the beta spectrum near its end point. So far the most successful ones of such experiments have been conducted with tritium [43] yielding only an upper limit. The effort still continues e.g. in the future KATRIN\(^1\) experiment in Karlsruhe, Germany.

4.1 Fourth-forbidden non-unique beta decays of $^{113}$Cd and $^{115}$In

The isotopes $^{113}$Cd and $^{115}$In share the unique feature that they both have a fourth-forbidden non-unique beta decay as their only decay channel, if the tiny ultra-low-$Q$-value decay channel of $^{115}$In with an extremely low branching ratio is neglected (Figure 4.1). Even though the theory for describing such decays has been ready and experimental data has been available for decades, our calculation applying the MQPM [40] was the first theoretical calculation for the half-lives and beta spectra of these nuclei. Later we recalculated these decays as the first application for the pnMQPM

\(^1\)Abbreviation of “KArlsruhe TRItium Neutrino experiment”.
The nucleus $^{115}$In is interesting also due to the possibility of applying its inverse beta decay for detecting solar neutrinos. This was first suggested in [45] and has been considered to be used in the LENS\textsuperscript{2} project [1].

The adopted valence space for our computations is illustrated in Figure 4.2. Both the initial and final state wave functions were dominantly of one-quasiparticle nature both in the MQPM and pnMQPM results, which was to be expected considering that they are ground states of the respective odd-mass nuclei. The low-energy spectra were reproduced reasonably well with both models with the exception of the $^{113}$Cd pnMQPM spectrum, in which the dominantly three-quasiparticle states remained notably higher in energy than their experimental counterparts. When compared to the experimental data (Figure 4.3), the $^{113}$Cd half-life seemed to notably improve when moving from the MQPM to the pnMQPM description. For $^{115}$In, there did not seem to be much improvement.

The situation got more interesting as the effect of the $Q$ value was investigated closer. We calculated the theoretical half-lives published in [40] and [44] and presented in Figure 4.3 with the $Q$ values taken from Nuclear Data Sheets [18, 20]. However, the half-life has a strong dependence on the $Q$ value and the agreement on the $Q$ value between different experiments is not as good as on the half-life. The $Q$ value dependence of the theoretical half-life is presented in Figures 4.4 and 4.5 along with the results of only those of the experiments that yielded both half-life and $Q$ value.

For $^{113}$Cd all but the latest one of these experiments agree better with the pnMQPM than with the MQPM result. Unfortunately, no ion-trap precision measurement for

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\textsuperscript{2}Abbreviation of “Low Energy Neutrino Spectroscopy”.

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\textbf{Figure 4.1:} Decay scheme for $^{113}$Cd and $^{115}$In. The data is compiled from Refs. [18, 20, 46, 47].
4.1 Fourth-forbidden non-unique beta decays of $^{113}$Cd and $^{115}$In

Figure 4.2: Schematic illustration of the chosen valence space for the $^{113}$Cd and $^{115}$In calculations. Approximate locations of the Fermi levels are indicated with $\epsilon_n$ and $\epsilon_p$ for neutrons and protons respectively.

Figure 4.3: Collected half-life measurements and calculations for both $^{113}$Cd (left) and $^{115}$In (right). The horizontal axis shows the year of publication, illustrating the development of experimental techniques. The experimental values are from [48] (Gre70), [49] (Mit88), [50, 51] (Ale94), [52] (Dan96), [3] (Goe05), [53] (Bel07) and [54] (Daw09) for cadmium and from [55] (Mar50), [56] (Ben61) and [57] (Pfe78) for indium.
**Figure 4.4**: Calculated $^{113}\text{Cd}$ half-life as a function of the decay $Q$ value. The experimental measurements are from [49] (Mit88), [50, 51] (Ale94), [52] (Dan96), [53] (Bel07) and [54] (Daw09).

**Figure 4.5**: Calculated $^{115}\text{In}$ half-life as a function of the decay $Q$ value. The experimental values are from [55] (Mar50), [36] (Bea61) and [57, 58] (Pfe78). The vertical dotted line represents the Penning-trap measurements [46, 47] for the $Q$ value. Their difference and their error bars are smaller than the line thickness.
4.1 Fourth-forbidden non-unique beta decays of $^{113}$Cd and $^{115}$In

![Image of beta spectra](image)

**Figure 4.6**: Calculated MQPM and pnMQPM beta spectra compared to the experimental one from [50] for $^{113}$Cd. Previously published in [59].

the $Q$ value has been published to date. In the case of $^{115}$In the pnMQPM result agrees perfectly with the latest experiment (labeled Pfe78), which in turn had a notably different $Q$ value than the previous measurements. The two fresh Penning trap experiments [46, 47] agree with Pfe78.

The experiments for the highly-forbidden beta decays are very demanding, because the half-lives are comparable to those of double beta decays. While all the different half-life measurements agree very well with each other (Figure 4.3), the beta spectrum shape especially on the low-energy end has been more challenging to determine experimentally. This can be seen clearly by comparing the three measured spectrum shapes for $^{113}$Cd (Figures 4.6, 4.7, 4.8 and 4.9). In each figure, the theoretical MQPM and pnMQPM spectra have been calculated using the $Q$ value obtained from the corresponding experiment.

Both experimental and theoretical spectra share the feature of a “hill”. In the theoretical spectra this hill is located at approximately 0.2 MeV, and in the experimental spectra around 0.15 MeV [52, 53, 54] or 0.2 MeV [50]. The low-energy behavior of the theoretical spectra cannot be accessed in the measurements since the experiments are inaccurate at low electron energies. The experimental spectra differ significantly from each other on the low-energy region since they have different statistics and the extrapolation procedure adopted is questionable.

The shape factor used in the fitted function for the experimental beta spectra of $^{113}$Cd is the one suited for the third-forbidden unique decay [60]. While this choice
Figure 4.7: Calculated MQPM and pnMQPM beta spectra compared to the experimental one from [52] for $^{113}\text{Cd}$. Previously published in [59].

Figure 4.8: Calculated MQPM and pnMQPM beta spectra compared to the experimental one from [53] for $^{113}\text{Cd}$. Previously published in [59].
4.1 Fourth-forbidden non-unique beta decays of $^{113}$Cd and $^{115}$In

**Figure 4.9:** Calculated MQPM and pnMQPM beta spectra compared to the experimental one from [54] for $^{113}$Cd.

**Figure 4.10:** Calculated MQPM and pnMQPM beta spectra compared to the experimental one from [57] for $^{115}$In. Previously published in [59].
is dubious, it seems to work reasonably well in the high-energy end of the spectrum. Since the uncertainties in the measurements are significantly larger in the low-energy end of the spectrum, the parameters of the fit are more sensitive to the details of the high-energy spectrum, resulting in the differences between the experiments seen in the low-energy region. Unfortunately, no simple fitted function with a reasonably small number of parameters is available for the non-unique case. Due to the dubious shape factor used in the experimental fits and the very different statistics on the low-energy region in different experiments, no definite conclusions can be drawn from the presently available data.

For $^{115}$In there is only one existing measurement for the spectrum shape [57] (Figure 4.10). The experimental spectrum hints to the existence of a hill around 0.2 MeV, whereas the calculations put the hill near 0.3 MeV. The spectrum shape for the region below 0.1 MeV is completely unknown. The lack of more precise data makes the comparison of the spectra ambiguous.

4.2 Single-beta decay channels of $^{96}$Zr

The zirconium isotope $^{96}$Zr decays primarily via double beta decay [61]. Although the single beta decay for this isotope is not yet experimentally observed, it is energetically allowed and hence competes with the double beta decay (See Figure 4.11). The experimental lower limit for the single-beta-decay half-life is $3.8 \times 10^{19}$ years (90\% C.L.) [62]. The single beta decay is hindered, because all the energetically allowed single-beta-decay channels are highly forbidden and the $Q$ values of these channels are very low.

The geochemical measurements for the double-beta-decay half-life of $^{96}$Zr have produced contradictory results: The experiment of Kawashima et al. [63], where a zircon mineral sample from Cable Sands, Australia, was analyzed to determine the excess amount of $^{96}$Mo, yielded the half-life of $(3.9 \pm 0.9) \times 10^{19}$ years. However, a similar experiment by Wieser et al. [64] on a zircon mineral from the same geographical region resulted in a significantly shorter estimate, $(9.4 \pm 3.2) \times 10^{18}$ years.

In addition to the geochemical experiments, the double-beta-decay half-life has also been measured in the NEMO-2 [65] and NEMO-3 experiments [66]. Their results, $(2.1^{+0.8}_{-0.4}(\text{stat.}) \pm 0.2(\text{syst.})) \times 10^{19}$ y and $(2.35 \pm 0.14(\text{stat.}) \pm 0.16(\text{syst.})) \times 10^{19}$ y respectively, are in perfect agreement with each other but disagree with both of the geochemical measurements.

The aim of our calculation [67] for the half-lives of the single-beta-decay channels was to estimate the possible contamination for the geochemical double-beta-decay half-
4.2 Single-beta decay channels of $^{96}$Zr

Figure 4.11: Decay scheme of $^{96}$Zr. The dotted decay channels have not yet been experimentally observed. The data on this figure is compiled from [62], [68] and [69]. Originally published in [70].

The calculated single-beta-decay half-life is well within the error bars of both the geochemical experiments. Therefore correcting for the beta-decay contamination is not crucial before more accurate geochemical data becomes available, especially since life measurements. As seen in Figure 4.11, the single beta decay of $^{96}$Zr is followed by another beta decay to $^{96}$Mo with a negligible time scale compared to the $^{96}$Zr lifetime, and it is impossible to know if a $^{96}$Mo atom in a geological sample has been produced via a double beta decay or two consecutive single beta decays.

In our calculation the valence space for both protons and neutrons consisted of 15 states reaching from the $3\hbar\omega$ to the $5\hbar\omega$ oscillator major shell (Figure 4.12). The procedure for describing the $^{96}$Nb daughter nucleus using the pnQRPA is described in Chapter 2. The interaction was unscaled in the pnQRPA calculation, i.e. the bare values $g_{ph} = g_{pp} = 1$ were used for each multipolarity in the pnQRPA.

Our calculated results for the partial half-lives imply that the decay channel to the $5^+$ excited state clearly dominates in the single-beta decay. According to our results, the decay to the $6^+$ ground state is completely negligible and the decay to the $4^+$ state only contributes a couple per cent. The total computed half-life, $2.4 \times 10^{20}$ y, is roughly an order of magnitude longer than the current experimental lower limit, hinting that this beta decay half-life might be reachable in near-future experiments. In [70], we have recomputed the half-lives using newer $Q$ values, but the updated results are close to the ones published in [67] and the conclusions are not affected.
the disagreement with each other and the NEMO experiments remains an unanswered question for the geochemical experiments on $^{96}$Zr.

4.3 Ultra-low-$Q$-value decay of $^{115}$In

The beta decay of $^{115}$In to the first excited state of $^{115}$Sn (see Figure 4.1) was first observed by Cattadori et al. [71]. They recognized it to possibly have the lowest observed $Q$ value and hence suggested the possibility of using this decay as an independent probe for the neutrino mass. The existence of this decay channel was confirmed and the half-life measurement refined in an experiment conducted in the HADES underground laboratory in Belgium [46].

In Penning trap measurements conducted by the JYFLTRAP group in the Department of Physics at the University of Jyväskylä the $Q$ value of this decay was discovered to be $(0.35 \pm 0.17)$ keV, roughly an order of magnitude lower than the previous record [46]. The $Q$ value was further refined to $(0.155 \pm 0.024)$ keV by a similar independent measurement in the Florida State University [47].

The ground-state-to-excited-state transition is a second-forbidden unique beta minus
4.3 Ultra-low-$Q$-value decay of $^{115}$In

![Diagram of $^{115}$In decay](image)

**Figure 4.13**: Simple one-quasiparticle interpretation for the lowest states of $^{115}$In and $^{115}$Sn can easily explain the observed angular momenta and parities. The first excited state of $^{115}$Sn is simply the excitation of the unpaired neutron to the $1d_{3/2}$ orbit. This interpretation is also consistent with the MQPM [40] and pnMQPM [44] calculations.

decay. For unique decays all but one of the nuclear matrix elements of the general beta-decay formalism vanish, and the calculation of the decay half-life is simplified enormously: The partial half-life of this decay channel is simply inversely proportional to the square of the nuclear matrix element $M$ (equal to $M_4$ in the notation of [40] and $A M_{K+1,K1}^{(0)}$ in the notation of [31]), or more formally,

$$ T_{1/2} = \frac{1}{M^2 f_K(w_0, Z_f, R)}, $$

where $f_K(w_0, Z_f, R)$, given in (3.13), is the phase-space integral depending only on the end-point energy $w_0$, the charge of the daughter nucleus $Z_f$ and the nuclear radius $R$, but not the details of the nuclear wave functions.

We used the pnMQPM to calculate the initial and final nuclear wave functions using the same values for the adjustable parameters as in our earlier work for the ground-state-to-ground-state decay channel [44]. The composition of both the initial and final state wave function implies that these both states are dominantly one-quasiparticle states. This interpretation is consistent with the naïve one-particle/one-hole description depicted in Figure 4.13.

Unexpectedly, the computed half-life as the function of the $Q$ value (Figure 4.14) has a significant disagreement with the experimental results. As the pnMQPM description
Figure 4.14: Computed partial half-life of the $^{115}\text{In}$ decay to the lowest excited state of $^{115}\text{Sn}$ as a function of the $Q$ value. The green band represents an ad hoc 30% uncertainty in the nuclear matrix element, giving a visual clue of how insensitive the curve is to the uncertainties in the nuclear description. This figure was published in [72].

reproduces the ground-state-to-ground-state transition half-life rather accurately, it is reasonable to expect a similar agreement here. However, to reach the 1σ limits of the most accurate measurement available (the HADES half-life measurement [46] combined with the Florida State University $Q$ value measurement [47]) the theoretical half-life curve would have to be multiplied by roughly a factor of 15. If this difference was solely due to the inaccuracy of the nuclear matrix element, it would have to be off by roughly a factor of 4. An error that large requires that either our interpretation of the simple structure of the final state is wrong or we are missing a crucial piece of the puzzle.

While we cannot rule out the possibility that the discrepancy between the theoretical prediction and the experimental data would not be due to a more exotic configuration of the final state, there is another possible explanation for it. There are several little-studied effects stemming from atomic electrons: The electron screening, the atomic overlap effect, the exchange effects and the final-state interactions. While these effects are negligible for the beta decays studied before this one, the fact that we are now on a new record-low-$Q$-value regime motivated us to take a closer look at them.

*The electron screening* corrections are traditionally estimated using the Rose prescription [37]. There also exists a more refined formula for taking them into account, namely the completely relativistic expression by Lopez and Durand [73]. Neither of these approximations is applicable to the ultra-low $Q$ values: For the case of $^{115}\text{In}$ they
break down so badly that they actually yield a large negative half-life.

*The atomic overlap effect* is caused by the fact that the electron states of the final atom are not exactly the same as those of the initial atom. This effect was studied by Bahcall [74] for allowed decays. According to his estimates, the decay is hindered by this effect the more the lower the $Q$ value. The case with the lowest $Q$ value he studied was the $^{241}$Pu beta decay with the $Q$ value of 21 keV. For that decay his estimates yielded a 2% hindrance in the decay half-life. However, his method breaks down for $Q$ values as low as a few hundred eV.

*The exchange effects* were also first estimated by Bahcall in [74]. He concluded that these effects would cause additional reduction in the decay rate, 2% in the case of $^{241}$Pu. However, the theoretical effort by Harston and Pyper [75] contradicts his estimates, concluding that the exchange effects may actually *enhance* the decay. Their calculation yielded a 7.5% enhancement on the $^{241}$Pu decay rate.

The molecular *final-state interactions* have only been studied for the beta decay of tritium [76]. In case of $^{115}$In the role of final-state interactions in the lattice is a completely uninvestigated territory: We do not know if the chemical bonds of the indium atoms in the sample introduce any non-negligible corrections to the ultra-low-$Q$-value decay.

It should be emphasized that even if the discrepancy did turn out to be due to an incorrect interpretation of the final state, the fact that the atomic contributions are not investigated for energies this low remains. Now that the advancements in experimental techniques have made it possible to access such extreme $Q$ values, the theory should be extended to cover this region as well.

The nuclear theory community seems to have lost interest in the atomic effects after the early studies showed them to be negligible for the usual beta-decay $Q$ values. An exception for this has been the tritium beta decay, which has been used as a tool in search for the neutrino mass (e.g. [43]), and where the atomic cloud is extraordinarily simple having only one electron. Now that the experimental techniques have been developed far enough to access decays where the old approximations break down, new theoretical work is needed to fill this gap in beta-decay theory. However, finding other likely candidates for observing an ultra-low-$Q$-value decay is currently difficult due to the fact that the atomic masses (or the ground-state-to-ground-state beta-decay $Q$ values) are not yet systematically measured with sufficient accuracy, not even in the valley of beta stability.

Experimental verification of the atomic contributions to beta decays is not possible at the moment either, because the uncertainties in the nuclear wave functions are still so large that the small corrections are dwarfed by their presence. Therefore properly closing all the open questions presented here may have to wait until the nuclear struc-
ture theory has advanced considerably. Still, theoretical estimates for the magnitude of the atomic contributions for the ultra-low-$Q$-value beta decays can be done. If they turn out to be large enough and if other experimentally observable cases were found, it might be possible to verify their existence experimentally.
5 Summary

In this thesis several rare beta decays were investigated using theoretical nuclear-structure and decay tools. These decays were the highly-forbidden non-unique beta decays of $^{113}\text{Cd}$ and $^{115}\text{In}$, the single-beta-decay channels of $^{96}\text{Zr}$ and the recently discovered ultra-low-$Q$-value decay channel of $^{115}\text{In}$.

As part of the project the proton-neutron microscopic quasiparticle-phonon model (pnMQPM) was developed. Its application to the ground-state-to-ground-state decays of $^{113}\text{Cd}$ and $^{115}\text{In}$ was successful. The advantages of the pnMQPM over MQPM are that one can use partly different set of experimental data to fine-tune the model parameters and that the pnQRP A fulfills the Ikeda sum rule, possibly improving the beta-decay calculations.

The primary weakness of the pnMQPM model is that its configuration space does not include the three-proton or three-neutron degrees of freedom. For some states these degrees of freedom are important and the pnMQPM wave functions are inevitably more inaccurate than the MQPM ones. This is a severe limitation of the pnMQPM. However, on the applications presented in this thesis, these contributions were not significant, since the initial and final states of the studied decays of the odd-mass nuclei were dominantly one-quasiparticle states.

When studying the wave function composition of the low-energy three-quasiparticle states, we also observed that in the MQPM picture there usually is one dominant configuration (a quasiparticle coupled to a QRP A phonon), but in the pnMQPM there are often four or five equally strong major configurations (a quasiparticle coupled to a pnQRP A excitation). In this sense, the MQPM approach is a more natural way of describing the structure of the wave functions.

The half-lives of the fourth-forbidden non-unique ground-state-to-ground-state beta decays of $^{113}\text{Cd}$ and $^{115}\text{In}$ were slightly better estimated using the pnMQPM than the MQPM approach, although a large difference was not to be expected, since all the involved states had only small three-quasiparticle components in both models. The beta spectra were very similar, and the experimental data is not accurate enough to favor one model over another in this respect.

Our investigation of the $^{96}\text{Zr}$ single-beta-decay channels using the pnQRP A approach yielded a half-life estimate of $2.4 \times 10^{20}$ y. This value is roughly an order of magnitude
longer than the current experimental lower limit for the single beta decay and well within the uncertainties of the geochemical results for the double-beta-decay half-life. Therefore the contamination from the single-beta-decay channels on the geochemical experiments can presently be neglected.

Our calculation on the 115In second-forbidden unique beta decay to the first excited state of 115Sn using the pnMQPM approach is off by roughly a factor of 15. Because the decay is unique, the dependence of the half-life on the nuclear wave functions is extraordinarily simple, and the inaccuracy of the model cannot explain the discrepancy unless our interpretation of the dominantly one-quasiparticle structure of the final state is completely wrong.

The $Q$ value of this decay is measured to be significantly lower than any beta-decay $Q$ value observed before, about an order of magnitude smaller than that of 187Re. This motivated us to consider an alternative explanation for the discrepancy: the atomic effects, namely the electron screening, the exchange and overlap effects and the molecular final-state interactions. Reviewing the literature on these effects revealed that the research on them is not directly applicable to this decay, but there is a clear trend that they grow more and more important as the $Q$ value decreases. These effects deserve to be studied more carefully.
References


REFERENCES


A Hamiltonian matrix elements for the pnMQPM

The one-quasiparticle states are eigenstates of the BCS Hamiltonian, i.e.

\[ \langle \alpha | H | \alpha' \rangle = \delta_{\alpha \alpha'} E_{\alpha} \]

(A.1)

for both proton and neutron states. The non-trivial Hamiltonian matrix elements are the ones involving a three-quasiparticle state as an initial or final state. In the case of the proton-odd nucleus they read

\[
\langle \pi | H | \nu \rangle = \frac{2 J_{\nu}}{J} \delta_{j_{\nu} j} \delta_{m_{\nu} m} \sum_{p' n'} \left[ (u_{p'} u_n X_{p'n'}^{\omega} - v_{p'} v_n Y_{p'n'}^{\omega}) u_p u_n G(pnp'n' J_{\omega}) \right. \\
- (v_{p'} v_n X_{p'n'}^{\omega} - u_{p'} u_n Y_{p'n'}^{\omega}) v_p u_n G(pnp'n' J_{\omega}) \\
\left. + (u_{p'} u_n X_{p'n'}^{\omega} + v_{p'} v_n Y_{p'n'}^{\omega}) u_p v_n F(pnp'n' J_{\omega}) \\
- (u_{p'} v_n X_{p'n'}^{\omega} + v_{p'} u_n Y_{p'n'}^{\omega}) u_p u_n F(pnp'n' J_{\omega}) \right], \tag{A.2}
\]

where \( G(pnp'n' J_{\omega}) \) and \( F(pnp'n' J_{\omega}) \) are the interaction matrix elements as defined in [14, 15], and

\[
\langle \nu' \omega'; jm | H | \nu \omega; jm \rangle = (E_n + E_{\omega}) \delta_{m m'} \delta_{\omega \omega'} \\
+ \frac{J_{\nu}}{J} J_{\nu'} \sum_{p} \left\{ \begin{array}{c}
\delta_{j_{\nu} j} \\
\delta_{j_{\nu'} j'}
\end{array} \right\} X_{p m}^{\omega} X_{p m'}^{\omega'} (E_n + E_{\omega'} - E_p) + \frac{\delta_{jj'} Y_{p n}^{\omega} Y_{p n'}^{\omega'} E_p \right), \tag{A.3}
\]

where the sum runs over the proton states. Correspondingly for the neutron-odd nucleus

\[
\langle \nu | H | \pi \omega; jm \rangle = \frac{2 J_{\nu}}{J} \delta_{j_{\nu} j} \delta_{m_{\nu} m} (-1)^{J_p + j_n - J_{\omega} + 1} \quad \\
\times \sum_{p' n'} \left[ (u_{p'} u_n X_{p'n'}^{\omega} - v_{p'} v_n Y_{p'n'}^{\omega}) v_p u_n G(pnp'n' J_{\omega}) \right. \\
- (v_{p'} v_n X_{p'n'}^{\omega} - u_{p'} u_n Y_{p'n'}^{\omega}) v_p u_n G(pnp'n' J_{\omega}) \\
\left. + (u_{p'} v_n X_{p'n'}^{\omega} + v_{p'} u_n Y_{p'n'}^{\omega}) v_p v_n F(pnp'n' J_{\omega}) \\
- (u_{p'} v_n X_{p'n'}^{\omega} + v_{p'} u_n Y_{p'n'}^{\omega}) u_p u_n F(pnp'n' J_{\omega}) \right]. \tag{A.4}
\]
Hamiltonian matrix elements for the pnMQPM

\[ \langle \pi' \omega'; jm | \hat{H} | \pi \omega; jm \rangle = (E_p + E_\omega) \delta_{pp'} \delta_{\omega \omega'} - (-1)^{j_p + j_p' + J_\omega + J_\omega'} \hat{J}_\omega \hat{J}_{\omega'} \times \sum_n \left\{ \begin{array}{ccc} j_p & j_n & J_\omega' \\ j_{p'} & j & J_\omega \end{array} \right\} X_{\pi \omega}^{j_p j_{\omega'}} X_{\omega'}^{j_{\omega} j_n} (E_\omega + E_{\omega'} - E_n) + \frac{\delta_{jj_n}}{j^2} Y_{\pi \omega}^{\omega'} Y_{\omega'}^{\omega} E_n \right\}, \] (A.5)

where the sum runs over the neutron states.
B Charge-changing transition densities in the pnMQPM

In the present notation the label $p$ (with and without a subscript) is always labeling a proton orbit and $n$ a neutron orbit. For the $\beta^-$ decay from a neutron-odd to a proton-odd nucleus, the pnMQPM charge-changing transition densities (CCTDs) are

$$(p_f||[c_p^\dagger c_n]_L||n_i) = \delta_{pp_f} \delta_{nn_i} \hat{L} u_p u_n, \quad (B.1)$$

$$(p_f||[c_p^\dagger c_n]_L||p_j \omega_i; j_i) = (-1)^{j_p + j_{\omega_i} + j_i + 1} \left[ \delta_{pp_f} \delta_{j_f j_i} (v_p u_n X_{pm}^\omega + u_p v_n Y_{pn}^\omega) \right. \\
+ \hat{J}_{\omega_i} \hat{L} \left. \left( \delta_{pp_p} \left\{ \begin{array}{ccc} j_f & L & j_i \\ J_{\omega_i} & j_n \end{array} \right\} v_p u_n X_{pf}^\omega - \delta_{pp_f} \frac{\delta_{jj_f} j_i}{j_f} u_p v_n Y_{pn}^\omega \right) \right], \quad (B.2)$$

$$(n_f \omega_f; j_f||[c_p^\dagger c_n]_L||n_i) = \hat{J}_{\omega_f} \hat{L} \left( \delta_{nn_f} \delta_{j_f j_i} (v_p u_n X_{pm}^\omega + u_p v_n Y_{pn}^\omega) \right. \\
+ \hat{J}_{\omega_f} \hat{L} \left. \left( \delta_{nn_p} \delta_{j_f j_i} u_p v_n X_{pf}^\omega - \delta_{nn_f} \frac{\delta_{jj_f} j_i}{j_f} v_p u_n Y_{pn}^\omega \right) \right], \quad (B.3)$$

and

$$(n_f \omega_f; j_f||[c_p^\dagger c_n]_L||p_j \omega_i; j_i)$$

$$= (-1)^{j_p + j_i + j_{\omega_i} + j_f} v_p v_n j_i j_f \hat{L} \left( \delta_{pp_p} \delta_{nn_f} \delta_{j_f j_i} \left\{ \begin{array}{ccc} j_f & L & j_i \\ J_{\omega_i} & j_n \end{array} \right\} X_{pf}^\omega X_{pn}^\omega - \delta_{jj_f} \frac{\delta_{jj_f} j_i}{j_f} Y_{pn}^\omega Y_{pf}^\omega \right) \\
+ \delta_{nn_f} \left\{ \begin{array}{ccc} j_f & L & j_i \\ J_{\omega_f} & j_n \end{array} \right\} \sum_{j_{n_i}} \left\{ \begin{array}{ccc} j_i & L & j_i \\ J_{\omega_f} & j_n' \end{array} \right\} X_{pf}^\omega X_{pn'}^\omega - \delta_{jj_f} \frac{\delta_{jj_f} j_i}{j_f} Y_{pn}^\omega Y_{pn'}^\omega \right) \\
+ \left\{ \begin{array}{ccc} j_{n_i} & L & j_f \\ J_{\omega_f} & j_n \end{array} \right\} X_{pf}^\omega X_{pn}^\omega + \delta_{jj_f} \frac{\delta_{jj_f} j_i}{j_f} Y_{pn}^\omega Y_{pn'}^\omega \right) \right]. \quad (B.4)$$
The pnMPQM $\beta^-$ CCTDs from a proton-odd to a neutron-odd nucleus are
\[
(n_f \mid [c_p^+ c_n^+]_L \mid p_i) = \delta_{pp_f} \delta_{nn_f} (-1)^{j_p + j_n + L} \hat{L} v_p v_n,
\]
\[
(B.5)
\]
\[
(n_f \mid [c_p^+ c_n^+]_L \mid n_i \omega_i; j_f) = (-1)^{j_n + j_f + L + 1} \hat{f} \left[ \delta_{n_f} \delta_{n_f} \delta_{p_f} \delta_{n_f} \right] \begin{pmatrix} j_f \\ j_n \\ J_{\omega_f} \\ j_p \end{pmatrix} v_p u_n \left( X_{p_f}^{\omega_f} X_{p_f}^{\omega_f} - \frac{\delta_{j_f} j_p}{j_f} u_p v_n Y_{p_f}^{\omega_f} \right) + \hat{J}_{\omega_f} L \left( \delta_{pp_f} \delta_{n_f} \delta_{p_f} \delta_{n_f} \right) \begin{pmatrix} j_f \\ j_n \\ J_{\omega_f} \\ j_p \end{pmatrix} u_p v_n \left( X_{p_f}^{\omega_f} X_{p_f}^{\omega_f} - \frac{\delta_{j_f} j_p}{j_f} u_p v_n Y_{p_f}^{\omega_f} \right),
\]
\[
(B.6)
\]
\[
(p_f \mid [c_p^+ c_n^+]_L \mid p_i \omega_i; j_i) = (-1)^{j_i + j_f + j_{\omega_f} + L + 1} u_p u_{\omega_f} \hat{f} \left[ \delta_{n_f} \delta_{n_f} \delta_{p_f} \delta_{n_f} \right] \begin{pmatrix} j_f \\ j_n \\ J_{\omega_f} \\ j_p \end{pmatrix} X_{p_f}^{\omega_f} X_{p_f}^{\omega_f} - \frac{\delta_{j_f} j_p}{j_f} Y_{p_f}^{\omega_f} Y_{p_f}^{\omega_f} + \delta_{n_f} \delta_{n_f} \delta_{p_f} \delta_{n_f} \left( \begin{pmatrix} j_f \\ j_n \\ J_{\omega_f} \\ j_p \end{pmatrix} X_{p_f}^{\omega_f} X_{p_f}^{\omega_f} + \frac{\delta_{j_f} j_p}{j_f} Y_{p_f}^{\omega_f} Y_{p_f}^{\omega_f} \right)
\]
\[
(B.7)
\]
and
\[
(p_f \mid [c_p^+ c_n^+]_L \mid n_i \omega_i; j_i)
\]
\[
= (-1)^{j_i + j_f + j_{\omega_f} + L + 1} u_p u_{\omega_f} \hat{f} \left[ \delta_{n_f} \delta_{n_f} \delta_{p_f} \delta_{n_f} \right] \begin{pmatrix} j_f \\ j_n \\ J_{\omega_f} \\ j_p \end{pmatrix} X_{p_f}^{\omega_f} X_{p_f}^{\omega_f} - \frac{\delta_{j_f} j_p}{j_f} Y_{p_f}^{\omega_f} Y_{p_f}^{\omega_f} + \delta_{n_f} \delta_{n_f} \delta_{p_f} \delta_{n_f} \left( \begin{pmatrix} j_f \\ j_n \\ J_{\omega_f} \\ j_p \end{pmatrix} X_{p_f}^{\omega_f} X_{p_f}^{\omega_f} + \frac{\delta_{j_f} j_p}{j_f} Y_{p_f}^{\omega_f} Y_{p_f}^{\omega_f} \right)
\]
\[
(B.8)
\]
The CCTDs for $\beta^+$ and electron capture decays from a neutron-odd to a proton-odd nucleus read
\[
(p_f \mid [c_p^+ c_n^+]_L \mid n_i) = \delta_{pp_f} \delta_{nn_i} (-1)^{j_p + j_n + L} \hat{L} v_p v_n,
\]
\[
(B.9)
\]
\[
(p_f \mid [c_p^+ c_n^+]_L \mid p_i \omega_i; j_f) = (-1)^{j_p + j_n + j_{\omega_f} + j_i + L + 1} \hat{f} \left[ \delta_{n_f} \delta_{n_f} \delta_{p_f} \delta_{n_f} \right] \begin{pmatrix} j_f \\ j_n \\ J_{\omega_f} \\ j_p \end{pmatrix} X_{p_f}^{\omega_f} X_{p_f}^{\omega_f} - \frac{\delta_{j_f} j_p}{j_f} Y_{p_f}^{\omega_f} Y_{p_f}^{\omega_f} + \hat{J}_{\omega_f} L \left( \delta_{pp_f} \delta_{n_f} \delta_{p_f} \delta_{n_f} \right) \begin{pmatrix} j_f \\ j_n \\ J_{\omega_f} \\ j_p \end{pmatrix} u_p v_n \left( X_{p_f}^{\omega_f} X_{p_f}^{\omega_f} - \frac{\delta_{j_f} j_p}{j_f} u_p v_n Y_{p_f}^{\omega_f} \right),
\]
\[
(B.10)
\]
\[
(n_f \omega_f; j_f \| [c_n^\dagger \tilde{c}_p]_L \| n_i) = (-1)^{j_p + j_n + L + 1} j_f \left[ \delta_{n,n_f} \delta_{J_n,L} (u_p u_n X_{p\gamma}^{\omega_f} + u_p v_n Y_{p\gamma}^{\omega_f}) \right. \\
+ \widehat{J}_{\omega_f} L \left( \delta_{mn_f} \left\{ \begin{array}{cc} j_f & L \\ j_p & J_{\omega_f} \end{array} \right\} v_p u_n X_{p\gamma}^{\omega_f} - \frac{\delta_{j_f j_p}}{j_f} u_p v_n Y_{p\gamma}^{\omega_f} \right) \right] 
\]

and

\[
(n_f \omega_f; j_f \| [c_n^\dagger \tilde{c}_p]_L \| P_i \omega_f; j_i) \\
= (-1)^{j_p + j_n + L + j_p + j_i + J_{\omega_f}} u_p u_n \hat{J}_{\omega_f} j_f \left[ \delta_{p,n_f} \delta_{o_o_f} \delta_{J_n,J_{\omega_f}} \left\{ \begin{array}{cc} j_f & L \\ j_p & J_{\omega_f} \end{array} \right\} X_{p\gamma}^{\omega_f} Y_{p\gamma}^{\omega_f} - \frac{\delta_{j_f j_p}}{j_f} u_p v_n Y_{p\gamma}^{\omega_f} \right] \\
\times \left[ \delta_{n,n_f} \left\{ \begin{array}{cc} j_f & j_i \\ j_p & j_n \end{array} \right\} \sum_{j_n'} \left\{ \begin{array}{cc} j_f & j_n' \\ j_p & j_n \end{array} \right\} X_{p\gamma}^{\omega_f} X_{p\gamma}^{\omega_f} \right. \\
+ \delta_{mn_f} \left\{ \begin{array}{cc} j_f & j_i \\ j_p & j_n \end{array} \right\} \sum_{j_n'} \left\{ \begin{array}{cc} j_f & j_n' \\ j_p & j_n \end{array} \right\} X_{p\gamma}^{\omega_f} X_{p\gamma}^{\omega_f} + \frac{\delta_{j_f j_i}}{j_f} u_p v_n Y_{p\gamma}^{\omega_f} \right] \right]. 
\]

Finally, the CCTDs for the $\beta^+$/EC decays for the transition from a proton-odd to a neutron-odd nucleus are

\[
(n_f \| [c_n^\dagger \tilde{c}_p]_L \| P_i) = \delta_{pp_f} \delta_{mn_f} \hat{L} u_p v_n, 
\]

\[
(n_f \| [c_n^\dagger \tilde{c}_p]_L \| n_i \omega_f; j_i) = (-1)^{j_p + j_n + j_f + j_i} \hat{J}_{\omega_f} \hat{L} \left[ \delta_{n,n_f} \delta_{J_n,L} (u_p v_n X_{p\gamma}^{\omega_f} + v_p u_n Y_{p\gamma}^{\omega_f}) \right. \\
+ \delta_{mn_f} \left\{ \begin{array}{cc} j_f & j_i \\ j_n & J_{\omega_f} \end{array} \right\} u_p v_n X_{p\gamma}^{\omega_f} - \frac{\delta_{j_f j_i}}{j_f} u_p v_n Y_{p\gamma}^{\omega_f} \right], 
\]

\[
(p_f \omega_f; j_f \| [c_n^\dagger \tilde{c}_p]_L \| P_i) = (-1)^{j_p + j_n + j_f + j_P + j_{\omega_f}} \hat{J}_{\omega_f} \hat{L} \left[ \delta_{p,p_f} \delta_{J_n,L} (v_p u_n X_{p\gamma}^{\omega_f} + u_p v_n Y_{p\gamma}^{\omega_f}) \right. \\
+ \delta_{pp_f} \left\{ \begin{array}{cc} j_f & j_i \\ j_n & J_{\omega_f} \end{array} \right\} v_p u_n X_{p\gamma}^{\omega_f} - \frac{\delta_{j_f j_i}}{j_f} u_p v_n Y_{p\gamma}^{\omega_f} \right]. 
\]
and

$$( p_f \omega_f; j_f \Vert \llbracket c_n^\dagger, \tilde{c}_p \rrbracket_L \Vert n_i \omega_i; j_i )$$

$$= (-1)^{j_p + j_n + j_p + j_i + J_c + 1} v_p v_n \tilde{j_j} \tilde{j_f} L \left( \delta_{pp} \delta_{nn} \delta_{\omega \omega} \left\{ \begin{array}{ccc} j_f & L & j_i \\ j_n & J_{\omega} & j_p \end{array} \right\} - \tilde{J}_{\omega_i} \tilde{J}_{\omega_f} \right)$$

$$\times \left[ \delta_{pp} \left\{ \begin{array}{ccc} j_f & L & j_i \\ j_n & J_{\omega} & j_p \end{array} \right\} \sum_{p'} \left( \left\{ \begin{array}{ccc} j_n & J_{\omega f} & j_i' \\ j_n' & J_{\omega f} & j_p' \end{array} \right\} X^{\omega_f}_{p' n} X^{\omega_i}_{p' n} - \frac{\delta_{j_i j_i'} \gamma^{\omega_f}_{p' n} \gamma^{\omega_i}_{p' n}}{\tilde{j_i}} \right) \right.$$  

$$+ \delta_{nn} \left\{ \begin{array}{ccc} j_f & L & j_i \\ j_n & J_{\omega} & j_p \end{array} \right\} \sum_{n'} \left( \left\{ \begin{array}{ccc} j_f & J_{\omega} & j_p' \\ j_n' & J_{\omega_f} & j_p' \end{array} \right\} X^{\omega_f}_{p n'} X^{\omega_i}_{p n'} - \frac{\delta_{j_f j_i} \gamma^{\omega_f}_{p n'} \gamma^{\omega_i}_{p n'}}{\tilde{j_f}} \right)$$

$$\left. \left. \left. + \left\{ \begin{array}{ccc} J_{\omega_f} & j_p & j_f \\ j_n & L & j_i \end{array} \right\} X^{\omega_f}_{p n} X^{\omega_i}_{p n} + \frac{\delta_{j_i j_i} \delta_{j_f j_n}}{\tilde{j_i} \tilde{j_f}} \gamma^{\omega_f}_{p n} \gamma^{\omega_i}_{p n} \right\} \right) \right].$$

(B.16)
Theoretical description of the fourth-forbidden non-unique $\beta$ decays of $^{113}\text{Cd}$ and $^{115}\text{In}$

by

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Theoretical description of the fourth-forbidden non-unique $\beta$ decays of $^{113}$Cd and $^{115}$In

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(Received 25 January 2006; published 3 May 2006)

The half-lives and log $ft$ values for the fourth-forbidden non-unique beta decays of the ground states of $^{113}$Cd and $^{115}$In were calculated using a transparent formulation for the $\beta^-$ transition amplitude. The microscopic quasiparticle-phonon model (MQPM) was used to calculate the initial and final states of the transitions. The corresponding wave functions were described as linear combinations of one- and three-quasiparticle configurations built in a realistic single-particle model space by using a realistic microscopic two-body interaction. The computed results for the log $ft$ values and half-lives are reasonably close to the available experimental data.

I. INTRODUCTION

The fourth-forbidden non-unique beta decays have log $ft$ values larger than 20 and half-lives around $10^{15}$ years. Only three isotopes having only this decay channel open are currently known: $^{50}$V, $^{113}$Cd, and $^{115}$In [1]. There are, however, highly-forbidden beta-decay transitions, e.g., in the decays of $^{48}$Ca and $^{96}$Zr which also decay by the double beta decay (see, e.g., Ref. [2]). In these nuclei the decay $Q$ value allows the $0^+$ ground state to decay to the $4^+$ (fourth-forbidden non-unique), $5^+$ (fourth-forbidden unique), and $6^+$ (sixth-forbidden non-unique) states in the daughter nuclei $^{48}$Sc and $^{96}$Nb. The decay of $^{48}$Ca was previously treated by using the nuclear shell model in Ref. [3].

The microscopic quasiparticle-phonon model (MQPM) [4] was developed to describe states of open-shell odd-$A$ nuclei. Thus it is suited to study of the decays of $^{113}$Cd and $^{115}$In. For both of these decays some experimental data exists: in addition to the half-lives, log $ft$ values and the excitation spectra evaluated in Refs. [6,7], there is also a very recent measurement for the half-life of $^{113}$Cd [1]. These data have been collected in Fig. 1.

The kinematical part of beta decay is well established in literature (e.g., in Ref. [8]). In the present work we apply the formulation of Ref. [8] for the non-unique $\beta^-$ decays by giving explicit expressions for the involved shape functions and nuclear matrix elements. Calculation of the single-particle matrix elements and the nuclear matrix elements makes use of our explicitly written charge-changing transition densities, which have to be computed by using a nuclear model. For the presently discussed odd-nuclei $Q$ the needed transition densities are easily computed using the formalism of the MQPM [4].

The MQPM provides an internally consistent, fully microscopic way of describing spherical (or nearly spherical) open-shell odd-$A$ nuclei. The same nucleon-nucleon interaction is used all the way from generating the quasiparticles and phonons to coupling them to three-quasiparticle configurations. Thus far only allowed beta decays have been considered in the MQPM framework, e.g., in Ref. [4] and [9]. However, since the forbiddenness of the beta decay does not affect the description of the initial and final nuclear states, the MQPM computed transition densities just have to be implemented in the general beta-decay framework of Ref. [8]. Still, the computation of the log $ft$ values for non-unique forbidden decays takes a lot more effort than for the allowed or unique-forbidden decays as there are either four (second-forbidden and higher) or six (first-forbidden) nuclear matrix elements to be considered, instead of just one or two. Non-unique first-forbidden beta decay was discussed earlier in Refs. [10–12] for odd-odd mother nuclei.

This article is organized as follows. In Sec. II we give the necessary theoretical background on the MQPM and non-unique forbidden beta decay. In Sec. III we apply the reviewed formalism to compute beta-decay log $ft$ values and half-lives for the $^{113}$Cd and $^{115}$In decays. In Sec. IV we summarize our results and draw the conclusions.

II. THEORETICAL BACKGROUND

A. Microscopic quasiparticle-phonon model

In the BCS approach [13] the ground state of an even-even nucleus is described as a superconducting medium where all the nucleons have formed pairs that effectively act as bosons. Formally the BCS ground state is defined as

$$|\text{BCS}\rangle = \prod_{\alpha>0} (u_{\alpha} - v_{\alpha} c^\dagger_{\alpha} c_{\alpha})(\text{CORE}),$$

where $|\text{CORE}\rangle$ represents the nuclear core (effective particle vacuum), $c^\dagger_{\alpha}$ is the particle creation operator and $u_{\alpha}$ and $v_{\alpha}$ are the unoccupation and occupation amplitudes, which are to be determined. The notation of Baranger [14] is adopted here for the quantum numbers of the single-particle states: the Roman letter $a$ includes the quantum numbers $n_a, l_a$ and $j_a$. The Greek letter $\alpha$ includes all the quantum numbers of $a$ and the magnetic quantum number $m_a$. The notation $\alpha > 0$ is interpreted as $m_\alpha > 0$. The time-reversed operator has been defined as $c^\dagger_{\alpha} = (-1)^{l_\alpha + m_\alpha} c^\dagger_{-\alpha}$, where $-\alpha = \{a, -m_\alpha\}$.

The BCS ground state (1) acts as a vacuum for quasiparticles. The creation and annihilation operators for quasiparticles are constructed via the Bogolyubov-Valatin quasiparticle transformation: $a^\dagger_{\alpha} = u_{\alpha} c^\dagger_{\alpha} + v_{\alpha} c_{\alpha}$, $a_{\alpha} = u_{\alpha} c_{\alpha} + v_{\alpha} c^\dagger_{\alpha}$. The BCS quasiparticles satisfy the anticommutation relation $[a^\dagger_{\alpha}, a_{\beta}] = \delta_{\alpha\beta}$ and are therefore fermions.
The occupation and unoccupation amplitudes can be solved by applying a variational procedure for minimizing the energy of the BCS ground state for protons and neutrons separately. Since the BCS ground state (1) lacks good particle number the variation is constrained by requiring the average proton and neutron numbers in the BCS ground state to correspond to the variation. This is achieved by discarding the eigenstates with zero eigenvalue. The procedure to overcome this difficulty and to transform to an ordinary eigenvalue equation forms an overcomplete basis. The procedure is described in more detail in Ref. [4]. In the procedure the set of the three-quasiparticle states also is described. It turns out that when minimizing the ground-state energy with the variational procedure, the terms $H_{20}$ and $H_{02}$ vanish. This means that the quasiparticle transformation drops a large part of the short-range residual interaction in noninteracting quasiparticles.

In practice, in the BCS calculations the monopole interaction matrix elements are often scaled so that the experimental pairing gaps are reproduced. The notation for these phenomenological scaling constants used here is $g^{(\pi)}_{pair}$ for protons and $g^{(\pi)}_{pair}$ for neutrons. The pairing gaps can be extracted from experimental data by using the linear approximation formulas [15]

$$\Delta_p(\frac{1}{2}X) = \frac{1}{2}(-1)^{Z+1}(S_p(\frac{1}{2}+1)X) - 2\Sigma_p(\frac{1}{2}X) + \Sigma_p(\frac{1}{2}+1X),$$

and

$$\Delta_n(\frac{1}{2}X) = \frac{1}{2}(-1)^{A-Z+1}(S_n(\frac{1}{2}+1)X) - 2\Sigma_n(\frac{1}{2}X) + \Sigma_n(\frac{1}{2}+1X),$$

where $S_p(\frac{1}{2}X)$ and $S_n(\frac{1}{2}X)$ are the neutron and proton separation energies of the $\frac{1}{2}X$ nuclide, respectively.

The logical next step in the quasiparticle framework is to consider the two-quasiparticle excitations, and hence, the terms $H_{22}, H_{40},$ and $H_{44}$ of the residual interaction (2). This can be done via the quasiparticle random phase approximation (QRPA) [14]. The QRPA excitations—or phonons—are created with the operator

$$Q^\dagger_{\omega} = \sum_{a\omega, j} [X^a_{\omega, a} A^\dagger_{a\omega}(J_{ao}M) - Y^\omega_{a\omega} \tilde{A}_{a\omega}(J_{ao}M)],$$

where $\omega$ stands for the angular momentum $J_{ao}$, the parity $\pi_\omega$ and the additional index $k_\omega$ which identifies the different excitations with the same angular momentum and parity. The summation is restricted so that double counting of pairs is avoided. The two-quasiparticle creation and annihilation operators are defined as $\tilde{A}_{ab}(J_{ao}M) = (1 + \delta_{ab}(-1)^{j_1-j_2-1/2}[a_1^\dagger a_2^\dagger])_{L},$ and its time-reversed hermitian conjugate $\tilde{A}_{ab}(J_{ao}M) = (-1)^{j_L-M} A_{ab}(J_{ao}, -M).$ The $X$ and $Y$ amplitudes can be solved from the QRPA matrix equation, see, e.g., Ref. [16].

The simplest way to describe the states of an odd-$A$ nucleus in the quasiparticle picture is to consider only the states where one quasiparticle is created on the BCS vacuum (the ground state of the even-even reference nucleus). To improve this description however, three-quasiparticle excitations should be included. The MQPM is a fully microscopic way to introduce the three-quasiparticle correlations and, consequently, to take the last two residual interaction terms of Eq. (2), namely, $H_{31}$ and $H_{13},$ into account.

The MQPM excitation operator is written as

$$\Gamma^j_{\uparrow}(jm) = \sum_n C^j_{n\delta_{jm}} + \sum_{a\omega} D^a_{\omega\omega}[a^\dagger_{a\omega} Q^\dagger_{\omega}]_{jm},$$

where the amplitudes $C^j_n$ and $D^a_{\omega\omega}$ are now to be determined. Use of the methods of Ref. [16] leads to a generalized eigenvalue equation

$$(A B) \begin{pmatrix} C \\ B^T A^* \end{pmatrix} = \Omega \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} C \\ B^* \end{pmatrix},$$

where the submatrix $n$ of the metric matrix is nondiagonal, as the three-quasiparticle basis states do not form an orthogonal set. Typically the set of the three-quasiparticle states also forms an overcomplete basis. The procedure to overcome this difficulty and to transform to an ordinary eigenvalue equation is described in more detail in Ref. [4]. In the procedure the submatrix $n$ is diagonalized and a complete set of basis states is achieved by discarding the eigenstates with zero eigenvalue.

The charge-changing transition densities (CCTD) for the $\beta^-$ decay in the MQPM framework are

$$(p||[c^{\dagger}_{\nu'} c_{\nu}])_{L}\|n) = \hat{L} u_{\nu'} \nu \delta_{\nu \nu'} \delta_{\nu \nu'}$$

and

$$(n||[c^{\dagger}_{\nu'} c'_{\nu'}])_{L}\|p) = \hat{L} v_{\nu'} \nu \delta_{\nu \nu'} \delta_{\nu \nu'} (-1)^{j_L+j_{\nu'}+L},$$

for transitions between one-quasiparticle states, and

$$(\omega p j ||[c^{\dagger}_{\nu'} c_{\nu}])_{L}\|n) = (-1)^{j_L+j_{\nu}+j_{\nu'}} \hat{J}_{ao} E_{\omega} \left[ \begin{array}{c} j_n \\ j_p \\ j_{\nu'} \\ j_{\nu} \\ j_{ao} \end{array} \right] \times X_{ao} u_{\nu'} \nu \sigma_{ao} \delta_{\nu \nu'} (-1)^{j_L+j_{\nu'}+L}$$

$$+ \frac{\delta_{\nu \nu'}}{j_{ao} E_{\omega}} Y_{ao} v_{\nu'} \nu \sigma_{ao} \delta_{\nu \nu'} \delta_{\nu \nu'},$$

for transitions between two-quasiparticle states, and

FIG. 1. Experimental data [6,7] on the decays of the ground states of $^{113}$Cd and $^{115}$In.
and

\[
(\alpha j || [c_p^\dagger \tilde{c}_{\nu}^\dagger]_L || p) \\
= -(1)^{j_\nu + j_\nu - L} (-1)^{j_\nu + j_\nu - j} \tilde{j}_{\nu} \tilde{L} \tilde{j}
\]

\[
\times X_{pp}^{\nu} v_p u_p \sigma_{pp}^{-1} \delta_{mn} (-1)^{j_\nu + j_\nu - L}
\]

\[
+ \frac{\delta_{j_\nu} \delta_{j_\nu}}{f^2} u_{nn} v_p \sigma_{nn}^{-1} \delta_{pp},
\]

(11)

for transitions between one-quasiparticle and quasiparticle-phonon excitations. Here we have defined \(\sigma_{mn} = \sqrt{1 + \delta_{mn}}\) and \(X_{aa}^{\nu} = X_{aa}^{\nu} - (-1)^{j_\nu + j_\nu - L} X_{aa}^{\nu}\). For transitions between two quasiparticle-phonon excitations, one obtains

\[
(\alpha j || [c_p^\dagger \tilde{c}_{\nu}^\dagger]_L || o' p') \\
= - \left[ (-1)^{j_\nu + j_\nu - j} \left\{ \begin{array}{ccc} j & j' & j' \\ j_\nu & j_\nu & j_\nu \end{array} \right\} \delta_{pp'} \right.
\]

\[
\times X_{pp'}^{\nu} v_{p'} u_p \sigma_{pp'}^{-1} \delta_{mn} (-1)^{j_\nu + j_\nu - L}
\]

\[
\left. + \frac{\delta_{j_\nu} \delta_{j_\nu}}{f^2} u_{nn} v_p \sigma_{nn}^{-1} \delta_{pp'}, \right)
\]

(12)

where

\[
K(oa' o'; j) = \tilde{j}_{a'} \tilde{L}_{a'} \sum_b \left[ \left\{ \begin{array}{ccc} j & j & j \\ j_\nu & j_\nu & j_\nu \end{array} \right\} X_{ba} \tilde{X}_{ba} \tilde{X}_{ba} \right.
\]

\[
- \frac{\delta_{j_\nu} \delta_{j_\nu}}{f^2} \tilde{X}_{ba} \tilde{X}_{ba} \tilde{X}_{ba} \sigma_{ba}^{-1} \sigma_{ba}^{-1}
\]

(13)

and the matrix element \((o p j || [c_p^\dagger \tilde{c}_{\nu}^\dagger]_L || o' n' j')\) is obtained by making the substitution \(u_a \rightarrow v_o\) and \(v_n \rightarrow -u_a\).

**B. Non-unique forbidden beta decay**

The general formulation to calculate non-unique forbidden beta decays is provided in Ref. [8]. In this section this formulation is presented in a streamlined way allowing easy application to \(\beta^-\) -decay calculations. Unlike in Ref. [8], where natural units were used, we will use SI units in this work.

When only the energy spectrum of the escaping electron is observed and the angular dependence has been integrated over, the probability of electron emission in the energy interval \(W_e\) to \(W_e + dW_e\) is

\[
P(W_e) dW_e = \frac{G_F^2}{(\hbar c)^3} \frac{1}{2\pi^2 \hbar} C(W_e) p_e c W_e (W_0 - W_e)^2
\]

\[
\times F_0(Z, W_e) dW_e,
\]

(14)

where \(G_F/(\hbar c)^3\) is the Fermi coupling constant, \(C(W_e)\) is the shape factor (discussed below), \(W_0\) is the endpoint energy of the beta spectrum, and \(F_0(Z, W_e)\) is the Fermi function. Furthermore, \(W_e\) is the energy and \(p_e\) the momentum of the emitted electron. The kinematical factors \(p_e c W_e (W_0 - W_e)^2\) arise from the available phase space for the emitted electron and antineutrino. The half-life becomes then

\[
t_{1/2} = \frac{\ln 2}{\int_{W_{e,c}} P(W_e) dW_e},
\]

(15)

where \(m_e\) is the mass of the electron.

The half-life of the decay can also be written in the form

\[
t_{1/2} = \kappa/C,
\]

where the constant

\[
\kappa = \frac{2\pi^2 \hbar \ln 2}{(m_e c^2)^3 G_F^2/(\hbar c)^3}
\]

(16)

has the value 6147 [17] and the unitless integrated shape factor is

\[
\tilde{C} = \int_{W_{e,c}}^w C(W_e) p_{ew} (w_0 - w_e)^2 F_0(Z, w_e) dW_e,
\]

(17)

where the electron-mass scaled quantities are \(w_0 = W_0/(m_e c^2), w_e = W_e/(m_e c^2), p = p_{ec}/(m_e c^2) = \sqrt{w_e^2 - 1}\), and \(F_0(Z, w_e)\) is the Fermi function. The reduced half-life, or log \(t_f\) value, is obtained by multiplying the half-life with the following unitless integrated Fermi function

\[
f = \int_{W_{e,c}}^w p_{ew} (w_0 - w_e)^2 F_0(Z, w_e) dW_e
\]

(18)

and taking a base-10 logarithm.

The shape factor \(C(w_e)\) can be obtained from [8]. It can be written in the form

\[
C(w_e) = (6.706 \times 10^{-6}) K \sum_{k+k=K+1} \lambda_{k_e} (w_e^2 - 1)^k - 1
\]

\[
\times (w_0 - w_e)^{2(k-1)} g^2 D_{kk,kc} A_K
\]

\[
+ \sum_{k+k=K+2} \lambda_{k_e} (w_e^2 - 1)^k - 1
\]

\[
\times (w_0 - w_e)^{2(k-1)} g^2 D_{kk,kc} B_K,
\]

(19)

where \(k_e, k_c\) are positive integers emerging from the partial wave expansion of the lepton wave functions. Their relation to the orbital angular momentum \(l\) of the leptons is

\[
k = \begin{cases} l & \text{for } j = l - \frac{1}{2} \\ l + 1 & \text{for } j = l + \frac{1}{2} \end{cases}
\]

(20)
where \( j \) is the total angular momentum of the lepton, obtained by coupling the orbital angular momentum with the lepton spin.

Leading contributions to the shape factor come from the smallest possible electron (\( k_e \)) and neutrino (\( k_\nu \)) partial waves satisfying \( k_e + k_\nu = K + 1 \) and \( k_e + k_\nu = K + 2 \) as indicated in (19). Hence, for the fourth-forbidden decay we have lepton orbital angular momenta corresponding up to \( l_e, l_\nu \leq 6 \) partial waves. In Eq. (19) we have used the scaling

\[
6.706 \times 10^{-6} = \left( \frac{m_e c^2 \cdot \text{fm}}{\hbar c} \right)^2 \tag{21}
\]

by assuming that all the nuclear matrix elements in \( \bar{A}_K \) and \( \bar{B}_K \) are given in units of \( (\text{fm})^K \).

The quantities \( \bar{A}_K \) and \( \bar{B}_K \) can be expressed in terms of kinematical factors and nuclear form factors \( F_{KLS}(q^2) \) [8], where \( K \) is the multipolarity of the involved transition operator. The form factors can be related to the nuclear matrix elements for small momentum exchange \( q^2 \) in the impulse approximation. In this case the most important contribution to the decay rate will arise from the form factors that are related to the minimal transferred angular momentum. Order-of-magnitude considerations done in Ref. [8] imply that the following four form factors:

\[
V_{K,K-1,1}^{(0)}, \quad V_{K,K0}^{(0)}, \quad A_{K,K1}^{(0)}, \quad A_{K+1,K1}^{(0)} \tag{22}
\]

will be the most important when calculating the beta decay rate. Above we have denoted

\[
V/\bar{A}_{KLS}^{(0)} \equiv V/\bar{A}_{KLS}(q^2 = 0). \tag{23}
\]

In order to express the form factors \( V/\bar{A}_{KLS}^{(0)} \) in terms of nuclear matrix elements we use the impulse approximation [8]. This leads to

\[
R^L V_{KLS}^{(0)} \rightarrow (-1)^{K-L} g_L \sqrt{M_{KLS}^{(0)}}, \tag{24a}
\]

and

\[
R^L A_{KLS}^{(0)} \rightarrow (-1)^{K-L+1} g_A \sqrt{M_{KLS}^{(0)}}, \tag{24b}
\]

where \( R \) is the nuclear radius. We define the following matrix elements for a fixed multipolarity \( K \):

\[
R^{K-1} V_{K,K-1,1}^{(0)} \rightarrow -g_V V_{M_{K,K-1,1}}^{(0)} \equiv -g_{V} M_{1}, \tag{24c}
\]

\[
R^K V_{K,K0}^{(0)} \rightarrow g_V M_{K,K0}^{(0)} \equiv g_{V} M_{2}, \tag{24d}
\]

\[
R^K A_{K,K1}^{(0)} \rightarrow -g_A A_{M_{K,K1}}^{(0)} \equiv -g_{A} M_{3}, \tag{24e}
\]

\[
R^K A_{K+1,K1}^{(0)} \rightarrow g_A A_{M_{K+1,K1}}^{(0)} \equiv g_{A} M_{4}, \tag{24f}
\]

\[
R^K V_{K,K0}(k_e, 1, 1, 1) \rightarrow g_V M_{K,K0}(k_e, 1, 1, 1) \equiv g_{V} M_{2}^{(k_e)}, \tag{24g}
\]

\[
R^K A_{K,K1}(k_e, 1, 1, 1) \rightarrow -g_A A_{M_{K,K1}}^{(0)}(k_e, 1, 1, 1) \equiv -g_{A} M_{3}^{(k_e)}. \tag{24h}
\]

For convenience, we adopt the linear combinations

\[
M_{\pm} = M_2 \pm \sqrt{\frac{K + 1}{K}} \frac{g_A}{g_V} M_3 \tag{25}
\]

and

\[
M_{-}^{(k_e)} = M_2^{(k_e)} - \sqrt{\frac{K + 1}{K}} \frac{g_A}{g_V} M_3^{(k_e)} \tag{26}
\]

The factors containing the nuclear matrix elements are then given by

\[
\tilde{A}_K = \frac{2K + 1}{K} \tilde{M}_2^2 + \frac{1}{(2k_e + 2)^2} \left[ (\tilde{a} Z)^2 M_{-}^{(k_e)} \right]^2 \nonumber
\]

\[
+ 2 (\tilde{a} Z) w_{\nu} M_{-}^{(k_e)} + (1 + w_{\nu}^2) M_3^2 \nonumber
\]

\[
- \frac{2 \gamma_{k_e}}{k_e w_{\nu} (2k_e + 2)^2} [ (\tilde{a} Z) M_{-}^{(k_e)} + w_{\nu} M_3^2 ] \nonumber
\]

\[
+ \frac{1}{(2k_e + 2)^2} (w_0 - w_{\nu}) M_3^2 \nonumber
\]

\[
- \frac{2}{2k_e + 1} \sqrt{\frac{K + 1}{K}} [ (\tilde{a} Z) \tilde{M}_1 M_{-}^{(k_e)} + w_{\nu} \tilde{M}_1 M_3 ] \nonumber
\]

\[
+ \frac{2}{2k_e + 1} \sqrt{\frac{K + 1}{K}} \gamma_{k_e} \tilde{M}_1 M_{-}^{(k_e)} \nonumber
\]

\[
- \frac{2}{2k_e + 1} \sqrt{\frac{K + 1}{K}} (w_0 - w_{\nu}) \tilde{M}_1 M_3 \nonumber
\]

\[
+ \frac{2}{(2k_e + 1)(2k_e + 1)} \gamma_{k_e} (w_0 - w_{\nu}) \times [ (\tilde{a} Z) M_{-}^{(k_e)} + w_{\nu} M_3^2 ] M_3 \nonumber
\]

\[
- \frac{2}{(2k_e + 1)(2k_e + 1)} \gamma_{k_e} (w_0 - w_{\nu}) M_{-}^{(k_e)} \nonumber
\]

\[
\nonumber
\]

and

\[
\tilde{B}_K = \frac{K + 1}{(2k_e - 1)(2k_e - 1)} \left[ M_2^2 + 2 \frac{g_A}{g_V} \frac{k_e - k_{\nu}}{K + 1} M_2 M_3 \nonumber
\]

\[
+ \frac{(k_e - k_{\nu})^2}{(K + 1)(g_A^2/g_V^2)} M_3^2 \right] + \left( \frac{g_A}{g_V} \right)^2 M_2^2 \tag{28}
\]

Here \( g_V \) and \( g_A \) are the usual vector and axial-vector coupling constants.

In Eq. (19) we have defined according to [8]

\[
D_{K,k_e} = \frac{1}{\sqrt{2}} \sqrt{(2K + 1)!} \frac{1}{\sqrt{(2k_e - 1)!}(2k_e - 1)!}, \tag{29}
\]

\[
\tilde{D}_{K,k_e} = \frac{1}{\sqrt{2}} \sqrt{(2K + 1)!} \frac{1}{\sqrt{(2k_e - 1)!}(2k_e - 1)!}, \tag{30}
\]

and

\[
\lambda_{k_e} = \frac{F_{2\nu - 1}(Z, w_{\nu})}{F_0(Z, w_{\nu})}. \tag{31}
\]
The quantity \( F_{k-1}(Z, w_e) \) is the generalized Fermi function [8]
\[
F_{k-1}(Z, w_e) = 4^{k-1} (2k_c)(k_c + y_k)(2k_c - 1)!! \times e^{2m} \left( \frac{2mR}{\hbar} \right)^{2m(k_c - k)} \times \left( \frac{\Gamma(y_k + iy)}{\Gamma(1 + 2y_k)} \right)^2,
\]
where \( y = (\alpha Z w_e)/(p_c c) \). The dimensionless factors \( y_k \) and \( \tilde{a} \) in Eq. (27) read
\[
y_k = \sqrt{k_c^2 - (\alpha Z)^2}
\]
and
\[
\tilde{a} = \frac{\alpha \hbar}{R m_c}.
\]

The scaled matrix element
\[
\tilde{M}_1 = \frac{\hbar c}{m_c c^2} M_1 = 386.2 \text{ fm} \times M_1
\]
has been defined in Eq. (27) in order that all the matrix elements \( M_1, M_2, M_3, \) and \( M_4 \) in Eq. (24) have the units \( \text{fm}^k \). When expressed in these units the numerical values of the matrix elements \( M_1 \) and \( M_2 - M_3 \) can be inserted in the quantities \( \hat{A}_K \) and \( \hat{B}_K \) of Eqs. (27) and (28). Then the remaining scaling factor \( (\text{fm})^k \) of the squared matrix elements goes into the numerical factor (21), raised to the power \( K \) in Eq. (19).

The needed nuclear matrix elements can be calculated from the expression
\[
\frac{\hbar}{V/\Lambda M_{KLS}^{(0)}} = \frac{1}{\sqrt{2J} + 1} \sum_{pn} \langle \psi_f | [c_{p,n} \hat{c}_{K} | \psi_i \rangle,
\]
where the single-particle transition densities \( \langle \psi_f | [c_{p,n} \hat{c}_{K} | \psi_i \rangle \) are obtained from the nuclear wave functions \( \psi_f \) and \( \psi_i \), i.e., from the nuclear model. The involved single-particle transition matrix elements \( \frac{\hbar}{V/\Lambda M_{KLS}^{(0)}} \) are given by
\[
\frac{\hbar}{V} m_{KLS}(pn) = \frac{1}{\sqrt{2J} + 1} \langle p | T_{KLS} | n \rangle,
\]
and
\[
\frac{\hbar}{\Lambda} m_{KLS}(pn) = \frac{1}{\sqrt{2J} + 1} \langle p | yS T_{KLS} | n \rangle.
\]

The transition operator \( T_{KLS} \) is given in the spherical tensor notation as
\[
T_{KLS} = \begin{cases} r^L Y_{LM} \delta_{JK}, & S = 0, \\ r^L \langle -1 \rangle^{j_1 + j_2} Y_{LM} \sigma_{JK}, & S = 1, \end{cases}
\]
where \( Y_{LM} \) is the usual spherical harmonic and \( \sigma \) the Pauli spin operator.

To evaluate the single-particle matrix elements we use the relativistic single-particle spinor wave function
\[
\phi_{njl}(r) = \frac{G_{njl}(r)}{F_{njl}(r)},
\]
where the large component \( G_{njl} \) is taken to be a solution of the non-relativistic Schrödinger equation. These solutions are written in the form.
\[
G_{njl}(r) = i^{J} \bar{g}_{nl}(r)[\chi_{l}^2]_{jm},
\]
where \( \chi_{l}^2 \) is the nonrelativistic spin-\( \frac{1}{2} \) spinor. The small component of the spinor wave function (40) is
\[
F_{njl}(r) = \frac{\sigma \cdot p}{2M_N c} G_{njl}(r),
\]
where \( M_N \) is the nucleon mass and \( p \) its momentum, to be interpreted as a derivative operator. Taking \( g_{nl}(r) \) to be harmonic-oscillator wave functions the small component can be evaluated analytically. The result for \( J = \frac{1}{2} \pm \frac{1}{2} \) is
\[
F_{njl}(r) = \frac{i^{J +1/2}}{2(2M_N c)^b} (\frac{1}{b})^{J + 1/2} \frac{\hbar c}{b} g_{nl} \left\{ \frac{1}{b} \frac{1}{2} \frac{1}{2} \right\} [Y_{l^m} \chi_{l}^2]_{jm},
\]
where \( b \) is the harmonic-oscillator size parameter.

In terms of the large and small components of Eq. (40) the involved single-particle matrix elements can be written as
\[
v m_{KL0}(pn) = \frac{1}{\sqrt{2J}} \langle \psi_f | T_{KL0} | \psi_i \rangle \langle \psi_f | T_{KL0} | \psi_i \rangle + (F_p | T_{KL0} | F_n). \quad (44a)
\]
\[
\lambda m_{KL1}(pn) = \frac{1}{\sqrt{2J}} \langle \psi_f | T_{KL1} | \psi_i \rangle \langle \psi_f | T_{KL1} | \psi_i \rangle + (F_p | T_{KL1} | F_n). \quad (44b)
\]
and
\[
v m_{KL1}(pn) = \frac{1}{\sqrt{2J}} \langle \psi_f | T_{KL1} | \psi_i \rangle \langle \psi_f | T_{KL1} | \psi_i \rangle + (F_p | T_{KL1} | F_n). \quad (44c)
\]
After some algebra the matrix elements can be rewritten as
\[
v m_{KL0}(pn) = i^{J + 1/2} \frac{1}{2} \langle \psi_f | T_{KL0} | \psi_i \rangle \langle \psi_f | T_{KL0} | \psi_i \rangle + (F_p | T_{KL0} | F_n).
\]
\[
\lambda m_{KL1}(pn) = i^{J + 1/2} \frac{1}{2} \langle \psi_f | T_{KL1} | \psi_i \rangle \langle \psi_f | T_{KL1} | \psi_i \rangle + (F_p | T_{KL1} | F_n).
\]
and

\[ v_{KL} (pn) = i_{\ell_p + \ell_n + L + 1} \frac{1 + (-1)^{\ell_p + \ell_n + L + 1} L_{\ell_p \ell_n}}{2 K} \]

\[ \times \left( j_p \frac{1}{2} j_n - \frac{1}{2} \{ K | 0 \} \right) \]

\[ \times \left[ A_{KL} (pn) + B_{KL} (pn) \right] (-1)^{K + \ell_p + \ell_n + 1/2} \]

\[ \times r^{L-1}_{\ell_p} \Delta (l_p | l_n) \]

\[ + A_{KL} (pn) - B_{KL} (pn) \right] (-1)^{K + \ell_p + \ell_n + 1/2} \]

\[ \times r^{L-1}_{\ell_p} \Delta (l_p | l_n) \].

The notation \( \Delta (l_1 l_2 L) \) denotes the triangular function for the coupling of the angular momenta \( l_1, l_2, \) and \( L \). Furthermore, the geometric factors are

\[ A_{KL} (pn) = \frac{\tilde{j}_n^2 + (-1)^{\ell_p + \ell_n + K} \tilde{j}_n^{L-1} K}{2^{K+1} (K+1)(2L+1)} (-1)^{K+1} \]

\[ \times (K - 1 - L0 \{ 1 - \delta_{K0} \}) \]

\[ B_{KL} (pn) = (-1)^{\ell_p + \ell_n + L - 2} L^{L-1} (K - 0 \{ 0 | 0 \}) \]

and the auxiliary orbital angular-momentum quantum number has been defined as

\[ I = \begin{cases} \frac{l + 1}{2}, & j = l + \frac{1}{2}; \\ \frac{l - 1}{2}, & j = l - \frac{1}{2}. \end{cases} \]

The involved radial factors are defined as

\[ r^{L-1}_{\ell_p} = k(b) \left( b^{-1}(r^{L-1}_{\ell_p})_{pn} + 2 \sqrt{n_p + j_p + 1}(r^{L-1}_{\ell_p})_{pn} \right) \]

\[ r^{L-1}_{\ell_n} = k(b) \left( b^{-1}(r^{L-1}_{\ell_n})_{pn} + 2 \sqrt{n_p + j_p + 1}(r^{L-1}_{\ell_n})_{pn} \right) \]

and

\[ r^{L-1}_{\ell_p} = k(b) \left( b^{-1}(r^{L-1}_{\ell_p})_{pn} + 2 \sqrt{n_p + j_p + 1}(r^{L-1}_{\ell_p})_{pn} \right) \]

\[ - 2 b^{-1} \sqrt{n_p + j_p + 1}(r^{L-1}_{\ell_p})_{pn} \]

\[ + 4 \sqrt{(n_p + j_p + 1)(n_p + j_p + 1)(r^{L-1}_{\ell_p})_{pn}} \]

where the basic radial integral reads

\[ (r^{L}_{\ell_p})_{pn} = \int_0^{\infty} g_{n_p,l_p}(r) r^{L} g_{n_p,l_p} r^{2} dr. \]

The \( k \) dependent matrix elements \( M^{(k)}_{\ell_1} \) and \( M^{(k)}_{\ell_2} \) are calculated just like \( M_2 \) and \( M_4 \) except that the Coulomb factor

\[ I(k_e, 1, 1, r) = \left( \frac{3 - 2k_e + 1}{2k_e + 1} \right)^2 \]

\[ \frac{2k_e + 1}{2} \]

\[ \left( \frac{3 - 2k_e + 1}{2k_e + 1} \right)^2 \]

\[ r > R \]

is appended to the integrand of the radial integral (50).

From the single-particle matrix elements of Eqs. (45a)–

\[ (45c) \]

we can write our final matrix elements \( M_1 \)–\( M_4 \) as

\[ M_1 = \tilde{J}_i^{-1} \sum_{pn} v^{M}_{K,K-1,i}(pn) (\Psi_f || c^\dagger_{\nu} \tilde{c}_{\nu} || \Psi_i) \]

\[ M_2 = \tilde{J}_i^{-1} \sum_{pn} v^{M}_{K,K0}(pn) (\Psi_f || c^\dagger_{\nu} \tilde{c}_{\nu} || \Psi_i) \]

\[ M_3 = \tilde{J}_i^{-1} \sum_{pn} A_{M_{K1,K}(pn)} (\Psi_f || c^\dagger_{\nu} \tilde{c}_{\nu} || \Psi_i) \]

\[ M_4 = \tilde{J}_i^{-1} \sum_{pn} A_{M_{K+1,K1}(pn)} (\Psi_f || c^\dagger_{\nu} \tilde{c}_{\nu} || \Psi_i) \]

III. NUMERICAL APPLICATION

The mean-field single-particle states were generated by a coulomb-corrected Woods-Saxon (WS) potential with the parametrization of Bohr and Mottelson [18]. The adopted valence space consisted of the \( 3 \hbar \omega \) and \( 4 \hbar \omega \) oscillator major shells augmented by the \( 0 \hbar \omega \) states from the \( 5 \hbar \omega \) major shell. Slight modifications (see Table I) were made to some of the WS single-particle energies. In this way the resulting BCS quasiparticle spectra for protons and neutrons would better correspond to those experimental low-energy states in the adjacent proton-odd and neutron-odd nuclei that could be reasonably assumed to be dominantly of one-quasiparticle character. The two-body interaction matrix elements used throughout the calculations were generated from the Bonn one-boson-exchange potential applying G-matrix techniques [19].

The interaction matrix elements involved in the BCS calculations were scaled by a constant \( g_{\hbar \omega}^{(p)} \) for the protons

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Nucleus & Orbital & Woods-Saxon & Adjusted \\
\hline
\( ^{113}\text{Cd} \) & \( \pi 1p_{3/2} \) & -11.14 & -10.0 \\
\( \pi 0g_{9/2} \) & -8.57 & -9.4 \\
\( \nu 0l_{7/2} \) & -8.66 & -9.5 \\
\( \nu 0h_{11/2} \) & -6.04 & -7.1 \\
\hline
\( ^{116}\text{Sn} \) & \( \pi 1p_{3/2} \) & -9.58 & -8.8 \\
\( \pi 1d_{5/2} \) & -11.10 & -9.4 \\
\( \nu 1d_{5/2} \) & -7.74 & -7.4 \\
\( \nu 0h_{11/2} \) & -6.83 & -7.2 \\
\hline
\end{tabular}
\caption{Woods-Saxon energies and the manually adjusted energies in units of MeV. The proton and neutron orbitals are denoted with \( \pi \) and \( \nu \), respectively.}
\end{table}
and $g_{\text{pair}}^{(n)}$ for the neutrons, so that the phenomenological proton and neutron pairing gaps were reproduced. The pairing gaps were calculated by Eqs. (4a) and (4b) using the experimental separation energies from Refs. [6,7,20–23]. The following values for the scaling constants were obtained: for $^{112}\text{Cd}$ $g_{\text{pair}}^{(p)} = 1.02$ and $g_{\text{pair}}^{(n)} = 0.93$, and for $^{116}\text{Sn}$ $g_{\text{pair}}^{(p)} = 1.13$ and $g_{\text{pair}}^{(n)} = 0.94$. Hence the scaling needed was quite small, indicating that the monopole part of the used $G$-matrix is, as such, suitable for pairing calculations.

The quasiparticle energy spectra were quite successful in reproducing the low-energy spectra of the relevant odd-mass nuclei in all the cases except $^{113}\text{Cd}$. In the case of $^{116}\text{Sn}$, due to the magic proton number $Z = 50$ and the fact that the isotope $^{115}\text{In}$ was to be modeled using this reference nucleus, a proton number $Z = 48$ had to be used instead to achieve a reasonable quasiparticle energy spectrum for protons. Without this little trick, no reasonable ordering of the one-quasiparticle states could be achieved. The proton-quasiparticle spectrum for $Z = 50$ would have, however, nicely agreed with the low-energy spectrum of $^{117}\text{Sb}$.

In the QRPA calculations the interaction matrix elements were scaled separately for each multipole, as tabulated in Table II. The scaling constants were taken to be $a_{\text{pp}}$ for the particle-hole part and $g_{\text{pp}}$ for the particle-particle part of the proton-neutron residual interaction, as discussed in Ref. [24]. In this way the lowest excitation energy of each multipole was brought as close to the experimental energy as possible. If this was not possible with a reasonable scaling, then a more complex structure of the state was assumed and the scaling constant was taken to be 1 in default of any better alternative.

The $0^+$, $2^+$, and $4^+$ triplet of excited states around 1.3 MeV was ignored when doing the QRPA calculation for $^{112}\text{Cd}$ because of their known two-phonon nature [25], which traces all the way back to the BCS quasiparticle spectrum, where the attempts to improve the ordering of the computed single-quasiparticle spectrum by slight adjustments to the single-particle energies failed. However, it is reasonable to expect that this has little effect on the ground-state wave function.

Interestingly, in the MQPM spectrum for $^{113}\text{In}$ there are plenty of low-energy states not seen in the experimental spectrum. In most of these states the dominant component is $2^+ \otimes 02g_{9/2}$. The fact that these states push themselves so low in the MQPM spectrum reduces slightly the dominance of the $02g_{9/2}$ one-quasiparticle state in the ground-state wave function. Consequently, this may partially explain why the calculated decay rate is slower than the corresponding experimental one.

In the MQPM spectrum of $^{115}\text{Sn}$ the ordering of the $2^+_1$ and $1^+_1$ is reversed relative to the experiment. The MQPM spectrum of $^{115}\text{In}$, on the other hand, seems to agree with the experimental one rather nicely, even if the density of states above 800 keV of excitation energy is lower in the MQPM spectrum. This is natural, since the excitations of five and more quasiparticles are absent from the MQPM.

Once the MQPM description of the nuclei was complete, the resulting wave functions were used to compute the charge-changing transition densities needed in the beta-decay-rate calculations. It was observed that, unsurprisingly, the transition densities $^{115}\text{In}: 3^+_2 \| \tilde{c} \nu_{0g_{9/2}} \| 2g_{9/2}$ and $^{115}\text{Sn}: 1^+_2 \| \tilde{c} \nu_{2g_{9/2}} \| 2g_{9/2}$ were clearly dominant in the corresponding decays. The nuclear matrix elements, the log $f_i$ values, and the half-lives were then calculated utilizing the formulas presented in Sec. II B. The obtained nuclear matrix elements are listed in Table III.

| TABLE II. Values of the interaction scaling constants used in the QRPA calculations: $a_{\text{ph}}$ is the scaling constant for the particle-hole pair and $g_{\text{pp}}$ for the particle-particle interaction matrix elements. The blank voids denote the default value 1. |
|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|
| $J^+$ | $^{112}\text{Cd}$ | $g_{\text{ph}}$ | $g_{\text{pp}}$ | $^{116}\text{Sn}$ | $g_{\text{ph}}$ | $g_{\text{pp}}$ |
| 0$^+$ | 1.07 | 0.87 | 0.78 | 0.86 |
| 1$^-$ | 0.52 | 0.71 | 0.62 | 0.75 |
| 2$^-$ | 0.72 | 0.89 | 0.66 | 0.56 |
| 3$^-$ | 0.82 | 0.89 | 0.75 | 0.66 |
| 4$^-$ | 0.89 | 0.89 | 0.66 | 0.56 |
| 5$^-$ | 0.93 | 0.93 | 0.75 | 0.66 |
| 6$^-$ | 1.05 | 1.05 | 0.87 | 0.78 |

TABLE III. Calculated beta-decay nuclear matrix elements in units of fm$^4$.

<table>
<thead>
<tr>
<th>$^{113}\text{Cd} \to ^{113}\text{In}$</th>
<th>$^{115}\text{In} \to ^{115}\text{Sn}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>2.6313</td>
</tr>
<tr>
<td>$M_2$</td>
<td>596.51</td>
</tr>
<tr>
<td>$M_3$</td>
<td>532.99</td>
</tr>
<tr>
<td>$M_4$</td>
<td>876.49</td>
</tr>
<tr>
<td>$M_{11}$</td>
<td>655.96</td>
</tr>
<tr>
<td>$M_{12}$</td>
<td>612.71</td>
</tr>
<tr>
<td>$M_{13}$</td>
<td>589.52</td>
</tr>
<tr>
<td>$M_{14}$</td>
<td>575.23</td>
</tr>
<tr>
<td>$M_{21}$</td>
<td>586.88</td>
</tr>
<tr>
<td>$M_{22}$</td>
<td>458.36</td>
</tr>
<tr>
<td>$M_{23}$</td>
<td>527.70</td>
</tr>
<tr>
<td>$M_{41}$</td>
<td>514.96</td>
</tr>
</tbody>
</table>
The computed log $f_t$ values 23.94 and 23.20 were obtained for the ground-state-to-ground-state decays of $^{113}$Cd and $^{115}$In, respectively. The corresponding experimental values [6,7] are 23.20(10) and 22.5, respectively. This implies that the transitions are somewhat faster in reality than obtained by the MQPM. However, the difference is not very large and the trend of the log $f_t$ values is correctly reproduced.

The half-lives calculated from the obtained log $f_t$ values were $4.95 \times 10^{16}$ yrs for the decay of $^{113}$Cd and $1.99 \times 10^{15}$ yrs for the decay of $^{115}$In. The corresponding experimental values are $7.7(3) \times 10^{15}$ yrs and $4.41(25) \times 10^{14}$ yrs, respectively. The recent measurement for $^{113}$Cd [1] gives $(8.2 \pm 0.2 \text{(stat.)}^{+0.2}_{-0.1} \text{(sys.)}) \times 10^{15}$ yrs.

**IV. SUMMARY AND CONCLUSIONS**

The mother and daughter nuclei of the fourth-forbidden non-unique beta decays of $^{113}$Cd and $^{115}$In were modelled using the microscopic quasiparticle-phonon model. The resulting low-energy excitation spectra of the involved mother and daughter nuclei corresponded rather well to the experimental spectra. This was considered as a test of the theoretical
framework since only the composition of the wave functions of the ground states of the involved nuclei were of interest to us. The computed ground-state wave functions were used to calculate the required transition densities for evaluation of the beta-decay half-lives and log $f/t$ values of the discussed transitions. In this step we applied the Behrens and Bühring formulation of the $\beta^-$ decay amplitudes for non-unique forbidden transitions. According to the available experimental data, the transitions seem to be faster than what was predicted by the calculations.

The difference between the experimental and theoretical values is, however, not as large as one could expect considering previous calculations [9] for the allowed beta decays in some medium-mass and heavy nuclei.

ACKNOWLEDGMENTS

This work was supported by the Academy of Finland under the Finnish Center of Excellence Program 2006-2011 (Nuclear and Accelerator Based Program at JYFL).

Erratum: Theoretical description of the fourth-forbidden non-unique $\beta$ decays of $^{113}\text{Cd}$ and $^{115}\text{In}$


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(Received 5 June 2007; published 6 July 2007)

DOI: 10.1103/PhysRevC.76.019901 PACS number(s): 21.60.Jz, 23.40.Hc, 27.60.+j, 99.10.Cd

We have found an error in the final step of our computations for the above article. It turned out that our code for evaluating the generalized Fermi function $F_{k_e-1}(Z, w_e)$ was inaccurate for values of $k_e$ much larger than 1. This inaccuracy affected only the final log $ft$ values and half-lives of the decays.

The corrected log $ft$ and half-life values are presented in Table I. All of them are closer to the known experimental values than the erroneous results were. Our conclusions in the above article are therefore not affected by this correction. The transitions still seem to be faster than predicted by calculations, although the difference between the experimental and theoretical values is now considerably smaller.

TABLE I. Corrected values for the half-lives and log $ft$ values compared to the previously published erroneous ones. The experimental values are from Refs. [1,2].

<table>
<thead>
<tr>
<th></th>
<th>$^{113}\text{Cd}$</th>
<th></th>
<th>$^{115}\text{In}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>log $ft$</td>
<td>$T_{1/2}$</td>
<td>log $ft$</td>
</tr>
<tr>
<td>Erroneous</td>
<td>23.94</td>
<td>4.95 x 10^{16} yr</td>
<td>23.20</td>
</tr>
<tr>
<td>Corrected</td>
<td>23.45</td>
<td>1.69 x 10^{16} yr</td>
<td>22.69</td>
</tr>
<tr>
<td>Experimental</td>
<td>23.20(10)</td>
<td>7.7(3) x 10^{15} yr</td>
<td>22.5</td>
</tr>
</tbody>
</table>

Theoretical half-life for beta decay of $^{96}$Zr

by

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Theoretical half-life for beta decay of $^{96}$Zr

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Received 5 December 2006
Published 30 March 2007
Online at stacks.iop.org/JPhysG/34/837

Abstract
Highly forbidden $\beta^-$ decay of the $0^+$ ground state of $^{96}$Zr is studied. Partial half-lives to the lowest $6^+$, $5^+$ and $4^+$ states of $^{96}$Nb have been computed using the proton–neutron quasiparticle random-phase approximation (pnQRPA) with realistic two-body interactions in a large single-particle space. We find that the decay is dominated by the unique fourth-forbidden transition to the $5^+$ state with the half-life 2.4 x $10^{20}$ y. This half-life is an order of magnitude longer than the one measured for the double beta decay of $^{96}$Zr.

1. Introduction

The general formalism to calculate the decay rates of beta-decay transitions is well known [1, 2]. The decay amplitudes for forbidden beta transitions involve complicated expressions containing various phase-space factors and nuclear matrix elements. The main problem in the evaluation of the transition rates has been the computation of the one-body transition densities contained in the nuclear matrix elements. For the light nuclei the nuclear shell model is the appropriate computational scheme [3–5], but for heavy nuclei it is easier to resort to the proton–neutron quasiparticle random-phase approximation (pnQRPA) [6] for odd–odd nuclei, or the microscopic quasiparticle-phonon model (MQPM) [2] for odd-$A$ nuclei.

The nuclei $^{48}$Ca and $^{96}$Zr share the same interesting feature: they both beta decay and double beta decay. In [5], it was found that the computed half-life of $^{48}$Ca was some 25 times longer than the measured [7, 8] double-beta-decay half-life. The double beta decay of $^{96}$Zr has been studied both experimentally and theoretically [9, 10]. The measured double-beta-decay half-life for $^{96}$Zr is $T_{\beta\beta}^{1/2} = 2.1^{+0.8}_{-0.4} \times 10^{19}$ y [10] which is comparable to the double beta half-life of $^{48}$Ca. Nuclear-structure aspects of double beta decay have been reviewed e.g. in [11].

The double beta decay of $^{96}$Zr is challenged by the sixth-forbidden beta transition to the $6^+$ ground state of $^{96}$Nb and the fourth-forbidden beta transitions to the first two excited states, $5^+$ and $4^+$ in $^{96}$Nb. The beta transition to the $5^+$ state is unique, i.e. only one nuclear matrix element is active in the corresponding decay amplitude. The decay scheme is shown in figure 1. All these transitions are hindered by their very small measured $Q$-values of 163 keV ($6^+$), 119 keV ($5^+$) and 17 keV ($4^+$). We use these values in our theoretical computation.
of the partial half-lives to these states. The unique fourth-forbidden transition is expected to dominate by $Q$-value and forbiddenness arguments.

Due to the very small $Q$-values the beta transitions are hard to measure. The current experimental half-life limit is $T_{1/2} > 3.8 \times 10^{19}$ y (90% CL) [12], not far from the measured double-beta-decay half-life. We discuss briefly the theoretical aspects of beta decay in section 2. In section 3 we summarize and discuss our results, and in section 4 we draw the conclusions.

2. Theoretical background

2.1. Formalism of forbidden beta decays

The general formulation of forbidden beta decays is provided in [1]. In this section, we use the presentation of [2] for the application to $\beta^-$-decay calculations. The half-life of beta decay can be written as

$$t_{1/2} = \frac{\ln 2}{\int_{W_0}^{\infty} P(W_e) dW_e},$$

where $m_e$ is the mass of the electron, $W_0$ is the endpoint energy of the beta spectrum and $P(W_e) dW_e$ is the probability of electron emission in the energy interval $[W_e, W_e + dW_e]$:

$$P(W_e) dW_e = \frac{G_F^2}{(\hbar c)^6} \frac{1}{2\pi^2\hbar} C(W_e) p_e c W_e (W_0 - W_e)^2 F_0(Z, W_e) dW_e,$$

where $G_F/(\hbar c)^3$ is the Fermi coupling constant, $C(W_e)$ is the shape factor given in detail in [2] and $F_0(Z, W_e)$ is the Fermi function. The kinematical factor $p_e c W_e (W_0 - W_e)^2$ arises from the available phase space for the emitted electron and antineutrino, and it contains the energy $W_e$ and momentum $p_e$ of the emitted electron.

The shape factor $C(W_e)$ contains the information about the nuclear structure through the various nuclear matrix elements that it contains. The involved four matrix elements can be
Theoretical half-life for beta decay of 96Zr written for Kth-forbidden beta-decay transitions \((K \geq 2)\) as

\[ M_1 = \hat{J}_i^{-1} \sum_{pn} V_{m_{K,K-1}}(pn) (\psi_f \| c_p^\dagger c_n^\dagger)_K \psi_i), \]
\[ M_2 = \hat{J}_i^{-1} \sum_{pn} V_{m_{K,0}}(pn) (\psi_f \| c_p^\dagger c_n^\dagger)_K \psi_i), \]
\[ M_3 = \hat{J}_i^{-1} \sum_{pn} \lambda_{m_{K,1}}(pn) (\psi_f \| c_p^\dagger c_n^\dagger)_K \psi_i), \]
\[ M_4 = \hat{J}_i^{-1} \sum_{pn} \lambda_{m_{K,1,1}}(pn) (\psi_f \| c_p^\dagger c_n^\dagger)_{K+1} \psi_i), \] (3)

where \(\hat{J}_i = \sqrt{2J_i + 1}\), \(J_i\) being the angular momentum of the initial state and the summation runs over all possible proton \((p)\) and neutron \((n)\) quantum-number combinations allowed by the single-particle transition matrix elements \(V/A m_{KLS}(pn)\). The corresponding transition operator is written in the spherical tensor notation as

\[ T_{KLSM} = \begin{cases} i^L r^L Y_{LM} \delta_{LK}, & S = 0, \\ i^L (-1)^{L+1-K} r^L [Y_L \sigma]_{KM}, & S = 1, \end{cases} \] (7)

where \(K\) is the tensorial rank of the transition operator, \(Y_{LM}\) is the usual spherical harmonic (orbital contribution) and \(\sigma\) is the Pauli spin operator (spin contribution). Detailed expressions for the single-particle transition matrix elements are given in [2].

In addition to the above listed matrix elements \(M_1 - M_4\) there are additional matrix elements involved, namely the matrix elements

\[ M_k^+, \quad M_l^-, \quad k_e = 1, 2, 3, 4, 5, 6, \] (8)

where \(k_e = 1 - 6\) for the decay to the 6\(^+\) state and \(k_e = 1 - 4\) for the decay to the 4\(^+\) state. These matrix elements are absent for the decay to the 5\(^+\) state since the involved transition is a unique transition involving only the matrix element \(M_4\). The \(M^+_k, M^-_l\) matrix elements can be obtained from the corresponding \(M^+_2, M^-_3\) matrix elements by adding a Coulomb factor to the involved radial integrals (see [2]).

The quantities \((\psi_f \| c_p^\dagger c_n^\dagger)_K \psi_i\), contained in the expressions for \(M_1 - M_4\), are the one-body transition densities involving the initial \((\psi_i)\) and final \((\psi_f)\) nuclear wave functions between which the transition proceeds. In this work, these transition densities are evaluated by the use of the pnQRPA many-body scheme discussed below.

2.2. Transition densities in the pnQRPA

In the pnQRPA formalism the one-body transition densities are given by

\[ \langle \omega_f \| [x_p^\dagger x_n^\dagger]_K \| 0^+_{g.s.} \rangle = \delta_{K,J_f} \hat{J}_f (u_p v_n X_{pn}^{\omega_f} + v_p u_n Y_{pn}^{\omega_f}), \] (9)

where the 0\(^+\) initial state is the ground state of \(^{96}\text{Zr}\) and at the same time the vacuum of the pnQRPA. The \(u\) and \(v\) amplitudes are the unoccupation and occupation amplitudes of the BCS, and the pnQRPA state \(\omega_f\) is written as [6, 13]

\[ |\omega_f \rangle = \sum_{pn} (X_{pn}^{\omega_f} [a_p^\dagger a_n^\dagger])_{J_f} + Y_{pn}^{\omega_f} [\tilde{a}_p \tilde{a}_n])_{J_f}) \rangle \text{pnQRPA}, \] (10)
where $|\text{pnQRPA}\rangle$ is the pnQRPA vacuum and the operators $\tilde{a}$ and $a^\dagger$ are tensor operators for the annihilation and creation of BCS quasiparticles. The amplitudes $X$ and $Y$ are the usual forward and backward going amplitudes of the pnQRPA and $\omega_f = (k, J_f)$ contains the angular-momentum quantum number $J_f$ and the index $k$ numbering the solutions of the pnQRPA equations. The sum runs over all proton and neutron single-particle orbitals which can be coupled to produce a proton–neutron pair state of positive parity and angular momentum $J_f$.

3. Results and discussion

The mean-field single-particle states used in our calculations were generated by a Coulomb-corrected Woods–Saxon (WS) potential with the parametrization of Bohr and Mottelson [14]. The adopted valence space was the same for protons and neutrons and consisted of the 1p-0f, 2s-1d-0g and 2p-1f-0h oscillator major shells. This set of 15 single-particle states is complete in the sense that it also contains the spin–orbit partner of any included orbital. The needed two-body interaction matrix elements were generated from the Bonn one-boson-exchange potential applying $G$-matrix techniques [15].

The interaction matrix elements involved in the BCS calculations were scaled by a constant $g^{(p)}_{\text{pair}}$ for the protons and $g^{(n)}_{\text{pair}}$ for the neutrons, so that the phenomenological proton and neutron pairing gaps were reproduced. The pairing gaps were calculated by using the linear approximation [16]

$$
\Delta_p(\frac{A}{2}X) = \frac{1}{4}(-1)^Z(2^+_1) - S_p(\frac{A+1}{2}X) - 2S_p(\frac{A-1}{2}X) + S_p(\frac{A}{2}X)$$  \tag{11}
$$
\Delta_n(\frac{A}{2}X) = \frac{1}{4}(-1)^{A-Z+1}(2^+_1) - S_n(\frac{A+1}{2}X) - 2S_n(\frac{A-1}{2}X) + S_n(\frac{A}{2}X),$$  \tag{12}

with the experimental separation energies taken from [17]. The values $g^{(p)}_{\text{pair}} = 0.974$ and $g^{(n)}_{\text{pair}} = 0.808$ were obtained for the pairing parameters.

In the pnQRPA calculation, two scaling parameters occur for the proton–neutron two-body matrix elements, namely the particle–hole parameter $g_{\text{ph}}$ and the particle–particle parameter $g_{\text{pp}}$ [6, 11, 18]. Traditionally, the particle–hole parameter has been used to fix the position of the Gamow–Teller giant $1^+$ resonance, and the particle–particle parameter has been fixed by the beta-decay data [6, 11, 19]. For the high-angular-momentum states ($4^+$, $5^+$ and $6^+$) of interest in this work such fixing method is not possible. Fortunately, the values of these parameters, if kept within reasonable range, do not affect much the beta-decay half-life of the $^{96}\text{Zr}$ decay. This is why we have decided to adopt the bare $G$-matrix values $g_{\text{ph}} = 1.0$ and $g_{\text{pp}} = 1.0$ for the present calculation.

In figure 2, we show our result for the computed energy spectrum of $^{96}\text{Nb}$. The above-mentioned values for the Hamiltonian parameters were used for all multipole states $J^\pi$. At first sight the correspondence between the theoretical and experimental spectra is not at all good. We have to bear in mind, however, that no adjustments of the level energies were done by varying the values of $g_{\text{pp}}$ separately for each multipolarity in the pnQRPA calculations. This was not needed since the beta-decay results are fairly insensitive to small variations in $g_{\text{pp}}$. Furthermore, figure 2 presents a blow-up of a narrow energy region of 0.87 MeV in an odd–odd nucleus. In fact, within this narrow interval, all the experimental levels, except the highest-lying $3^-$ state, have a theoretical counterpart. Two theoretical levels lack an experimental counterpart: the predicted $1^+$ and $8^+$ levels. It is well known that the level density in odd–odd nuclei is high, and thus these nuclei offer a considerable challenge for any theory framework trying to predict fine details of their energy spectra. Against this
Theoretical half-life for beta decay of $^{96}$Zr

Figure 2. Experimental and theoretical low-energy spectra of $^{96}$Nb. The data is from [20].

Table 1. Calculated beta-decay nuclear matrix elements for the three final states. Note the scaling indicated in the first line.

<table>
<thead>
<tr>
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<td>$M_2^{(1)}$</td>
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<td>$M_3^{(6)}$</td>
<td>4.423</td>
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</tr>
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</table>

background the computed spectrum stands out as reasonable. Finally, it has to be pointed out that the multipole assignments in the experimental spectrum are not completely certain.

We summarize our computed beta-decay results in tables 1 and 2. In table 1 we give the values of the matrix elements in (3)–(8) for the three different final states. Major
single-particle components of these transitions are the following: for the decay to the 6+ state the leading transition is the $\nu_0g_{7/2} \rightarrow \pi_0g_{9/2}$ neutron-to-proton transition. Also the transition $\nu_0h_{9/2} \rightarrow \pi_0f_{7/2}$ plays a role. For the unique beta transition to the 5+ state the most notable single-particle transitions are $\nu_2s_{1/2} \rightarrow \pi_0g_{9/2}$ and $\nu_1d_{5/2} \rightarrow \pi_0g_{9/2}$. The beta transition to the 4+ state is dominated by the $\nu_0g_{7/2} \rightarrow \pi_0g_{9/2}$ single-particle transition.

Table 2 displays our results for the partial half-lives. As can be seen the unique fourth-forbidden transition to the 5+ state dominates the beta decay and the total half-life assumes the value $T_{1/2}^{\beta} = 2.4 \times 10^{20}$ y. This is an order of magnitude longer than the measured half-life of the double beta decay of $^{96}\text{Zr}$. The 4+ and 5+ states in $^{96}\text{Nb}$ decay by fast gammas to the 6+ ground state which, in turn, $\beta^-$ decays to $^{96}\text{Mo}$ with a half-life of 23.35 h [20]. This means that the total decay half-life of the complete $\beta^-$ decay route $^{96}\text{Zr} \rightarrow ^{96}\text{Nb} \rightarrow ^{96}\text{Mo}$ is the previously stated $T_{1/2}^{\beta} = 2.4 \times 10^{20}$ y.

In principle, the beta decay of $^{96}\text{Zr}$ to $^{96}\text{Nb}$ and its subsequent fast decay to $^{96}\text{Mo}$ could be a contaminant in the geochemical measurement of the double-beta half-life of $^{96}\text{Zr}$. The geochemical measurement [21] gives the result $T_{1/2}^{\beta\beta} = (3.9 \pm 0.9) \times 10^{19}$ y. This means that the beta decay would produce an effect which is within the error limits of the geochemical measurement. Thus, the geochemical data should become more accurate before it makes sense to correct the corresponding half-life for the beta-decay contamination.

4. Summary and conclusions

We have computed the half-life of the beta decay of $^{96}\text{Zr}$ by using a realistic nuclear Hamiltonian and a large single-particle model space. According to our calculations the decay proceeds mainly as a unique fourth-forbidden transition to the 5+ state in $^{96}\text{Nb}$. The estimated half-life for the $^{96}\text{Zr} \rightarrow ^{96}\text{Nb} \rightarrow ^{96}\text{Mo}$ chain of beta decays is found to be roughly an order of magnitude longer than the measured lower limit of the half-life and an order of magnitude longer than the half-life of the $^{96}\text{Zr} \rightarrow ^{96}\text{Mo}$ double-beta decay transition. This leads to the conclusion that beta decay does not significantly contaminate the geochemical measurements of the double-beta half-life of $^{96}\text{Zr}$.

Acknowledgments

This work was supported by the Academy of Finland under the Finnish Centre of Excellence Programme 2006–2011 (Nuclear and Accelerator Based Programme at JYFL). We also thank the EU ILIASS project under the contract RII3-CT-2004-506222.

References

Theoretical half-life for beta decay of $^{96}$Zr

Microscopic quasiparticle-phonon description of beta decays of $^{113}$Cd and $^{115}$In using proton-neutron phonons

by

M.T. Mustonen and J. Suhonen


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Microscopic quasiparticle–phonon description of beta decays of $^{113}$Cd and $^{115}$In using proton–neutron phonons

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Received 26 June 2007; received in revised form 3 September 2007; accepted 19 September 2007

Available online 4 October 2007

Editor: J.-P. Blaizot

Abstract

The fourth-forbidden non-unique ground-state-to-ground-state beta decays of $^{113}$Cd and $^{115}$In are calculated using a realistic microscopic two-body interaction and a realistic single-particle model space. To describe the involved initial and final nuclear states we introduce a proton–neutron variant of the microscopic quasiparticle–phonon model (MQPM), the proton–neutron MQPM (pnMQPM). The states of the pnMQPM are created by coupling quasiparticles with phonons of the proton–neutron quasiparticle random-phase approximation (pnQRPA). The computed half-lives and log $f_I$ values are found to be in excellent agreement with experimental data. Computed beta spectra of the decays are also given.

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PACS: 21.60.Jz; 23.40.Hc; 27.60.+j

Keywords: Fourth-forbidden non-unique beta decay; Beta transition half-lives; Proton–neutron microscopic quasiparticle–phonon model

There are three currently known isotopes that decay by a fourth-forbidden non-unique beta transition. These are $^{50}$V, $^{113}$Cd and $^{115}$In. We recently studied the latter two decays using the microscopic quasiparticle–phonon model (MQPM) in Ref. [1]. In addition, there are highly-forbidden beta-decay transitions in $^{48}$Ca and $^{96}$Zr. These nuclei decay also by the double beta decay [2]. These two cases have been treated by the nuclear shell model and the proton–neutron quasiparticle random-phase approximation (pnQRPA), respectively, in Refs. [3,4].

The MQPM [5,6] is an internally consistent, fully microscopic method of determining the structure of medium-heavy or heavy spherical or nearly spherical odd-mass nuclei. It treats all the three parts of the Hamiltonian—the quasiparticle, phonon and quasiparticle–phonon terms—on equal footing. In the MQPM, QRPA phonons are coupled with BCS quasiparticles to form the three-quasiparticle states. In this Letter we introduce an alternative approach: we couple proton–neutron QRPA (pnQRPA) phonons with quasiparticles to form the three-quasiparticle states. The corresponding theoretical framework we call the proton–neutron MQPM (pnMQPM).

Since the pnQRPA satisfies the Ikeda sum rule [7], the pnMQPM approach can be expected to be more suitable for beta-decay calculations than the original MQPM framework. We first briefly introduce the basic theoretical elements of the pnMQPM, and as its first application we calculate here the half-lives and log $f_I$ values for the extremely slow beta decays of the ground states of $^{113}$Cd and $^{115}$In. We compare the obtained results with the available data and the results of the MQPM.

Application of both the MQPM and pnMQPM consists of four steps. As the first step, BCS [8] calculations are performed in the even–even reference nucleus adjacent to the odd-mass nucleus one aims to describe. The quasiparticles are defined via the Bogolyubov–Valatin transformation

\[
\begin{align*}
\{a^\dagger_a\} &= \{u_a c^\dagger_a\} + \{v_a \tilde{c}^\dagger_a\}, \\
\{a_a\} &= \{u_a c_a\} - \{v_a \tilde{c}_a\},
\end{align*}
\]  

where the occupation amplitudes $u_a$ and $v_a$ are solved via a variational procedure [9]. The notation of Baranger [10] is adopted for labeling the single-particle states: the Roman index $a$ includes the quantum numbers $n_a$, $l_a$ and $j_a$, and the cor-

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responding Greek index \( \alpha \) contains, in addition, the magnetic quantum number \( m_\alpha \). The operator \( \hat{c}^\dagger_{\alpha p} \) is the particle creation operator and \( \hat{c}_{\alpha p} \) is the time-reversed particle annihilation operator.

After the quasiparticle transformation the residual nuclear Hamiltonian can be written [9] as

\[
H_{\text{RES}} = H_{22} + H_{31} + H_{13} + H_{40} + H_{04},
\]

where each term \( H_{mn} \) is proportional to a normal-ordered product of \( m \) quasiparticle creation and \( n \) quasiparticle annihilation operators. The parts \( H_{22}, H_{40} \) and \( H_{04} \) in (2) are taken into account by the second step leading to the MQPM and pnMQPM.

It is here where the pnMQPM procedure deviates from that of the MQPM. In the MQPM the phonons of the model were calculated in the even–even reference nucleus via the QRPA procedure, whereas in the pnMQPM we use the pnQRPA to produce the phonons in the odd–odd isobars adjacent to the even–even reference nucleus.

In the pnQRPA the nuclear states are described as linear combinations of proton–neutron quasiparticle excitations, created through the operator

\[
Q^\dagger_{\omega}(jm) = \sum_{pn} \left( X^\omega_{pn}[a_p^\dagger a_n^\dagger]_J_{JM} + Y^\omega_{pn}[a_p a_n]_J_{JM} \right),
\]

where the index \( p \) runs over all the proton states and \( n \) over all the neutron states in the valence space. The subscript superscript \( \omega \) stands for the angular momentum \( J_\omega \), the parity \( \pi_\omega \), and the additional index \( k_\omega \), enumerating the different excitations with the same angular momentum and parity. The amplitudes \( X^\omega_{pn} \) and \( Y^\omega_{pn} \) can be solved from the pnQRPA matrix equation [9].

As the third step, we set up the basis for the states of the odd-mass nucleus we wish to describe. If the nucleus is proton-odd (neutron-odd), this set consists of proton (neutron) quasiparticle states plus neutron (proton) quasiparticles coupled with (low-lying) pnQRPA phonons. The basic pnMQPM excitation operator therefore reads in the case of a proton-odd nucleus

\[
G^\dagger_i (jm) = \sum_p C_p^i a_{pjm} + \sum_{n\omega} D^i_{n\omega}[a_n^\dagger Q^\omega_{\omega}]_{jm},
\]

where the first summation runs over those proton orbits that have the total angular momentum \( j \) and the second summation over those neutron–phonon pairs that can be coupled to \( j \). The index \( i \) enumerates the different pnMQPM states with the same \( j \). Similarly, in the case of a neutron-odd nucleus the operator is

\[
G^\dagger_i (jm) = \sum_n C_n^i a_{njm} + \sum_{p\omega} D^i_{p\omega}[a_p^\dagger Q^\omega_{\omega}]_{jm}.
\]

The constants \( C_p^i \) and \( D^i_{n\omega} \) (or \( C_p^i \) and \( D^i_{p\omega} \)) are to be solved from the equations of motion for the excitation operators.

The fourth and final step is to form the pnMQPM matrix through the equations-of-motion method [9,11] and solve its eigenstates. This step diagonalizes the parts \( H_{31} \) and \( H_{13} \) of the residual Hamiltonian (2). The set of basis states constructed in the previous step is usually overcomplete and the three-quasiparticle states are not generally orthogonal to each other.

In fact, we obtain the non-zero overlap

\[
\langle [a_n^\dagger Q^\omega_{\omega}]_{jm} | \tilde{H} [a_n^\dagger Q^\omega_{\omega}]_{jm} \rangle = \delta_{nn'} \delta_{\omega\omega'} + K(p)(\eta\omega'; j)
\]

in the case of a proton-odd nucleus or

\[
\langle [a_p^\dagger Q^\omega_{\omega}]_{jm} | \tilde{H} [a_p^\dagger Q^\omega_{\omega}]_{jm} \rangle = \delta_{pp'} \delta_{\omega\omega'} + (-1)^{j + j_p' + j_\omega + j_\omega'} K(q)(\eta\omega'; j)
\]

in the case of a neutron-odd nucleus. The auxiliary expressions \( K(p)(\eta\omega'; j) \) and \( K(q)(\eta\omega'; j) \) are defined as

\[
K(p)(\eta\omega1n_{202}'; j) = \frac{j_1 j_p}{j^2} Y^{\omega1}_{p_{1n}} Y^{\omega1}_{p_{1n}} \quad \text{and}
\]

\[
K(q)(\eta\omega1p_{2n}'; j) = \frac{j_1 j_n}{j^2} Y^{\omega1}_{p_{2n}} Y^{\omega1}_{p_{2n}}
\]

The vacuum \(| \rangle \) above is the pnQRPA vacuum.

The procedure for solving a generalized eigenvalue problem in an overcomplete basis is described in detail in Ref. [6] in the context of the MQPM. This partly removes also the difficulties with the Pauli exclusion principle which already start at the pnQRPA level when the equations of motion for the phonon states are solved using the quasiboson approximation [9]. For reasonable strengths of the two-body interactions this approximation introduces only small violations of the Pauli principle, as discussed, e.g., in [12]. A further violation of this principle is introduced when combining different phonons with the quasiparticles in the pnMQPM wave functions (4) and (5). Such a procedure introduces extra (spurious) states into the pnMQPM basis to be diagonalized. This overcompleteness and partly the extra violation of the Pauli principle can be cured for in the pnMQPM by the method of diagonalizing the overlap matrix (6) or (7). This method seems to work well since no extra states are seen in the computed low-energy spectra when compared to data.

For the needed Hamiltonian matrix elements we obtain

\[
\langle [a_n^\dagger Q^\omega_{\omega}]_{jm} | \tilde{H} [a_n^\dagger Q^\omega_{\omega}]_{jm} \rangle = (E_n + E_{\omega}) \delta_{nn'} \delta_{\omega\omega'} + \frac{j_1 j_p}{j^2} Y^{\omega1}_{p_{1n}} Y^{\omega1}_{p_{1n}} \times (E_\omega + E_{\omega'} - E_p) + \delta_{pp'} \delta_{\omega\omega'} Y^{\omega1}_{p_{1n}} Y^{\omega1}_{p_{1n}} F_p.
\]
\[ \langle \pi \mid \hat{H} \mid \pi Q^\dagger \rangle_{jm} \]

\[ = \frac{1}{j} \sum_{p,n'} \left[ \left( u_{p'} u_n X_{p'}^{q} - v_{p'} v_n Y_{p'}^{q} \right) u_{p} u_n G(pnp'n' J_{ao}) \right. \]

\[ + \left( v_{p'} v_n X_{p'}^{q} - u_{p'} u_n Y_{p'}^{q} \right) v_{p} u_n F(pnp'n' J_{ao}) \]

\[ + \left( v_{p'} u_n X_{p'}^{q} + u_{p'} v_n Y_{p'}^{q} \right) v_{p} u_n F(pnp'n' J_{ao}) \]

\[ - \left( u_{p'} v_n X_{p'}^{q} + u_{p'} v_n Y_{p'}^{q} \right) v_{p} u_n F(pnp'n' J_{ao}) \]

\[ \times \left[ \delta_{jm} + \frac{j}{2} \left( Y_{p'n'}^n X_{p'n'}^n (E_{ao} + E_{ao'} - E_n) \right) \right. \]

\[ \left. + \frac{\delta_{jm}}{j^2} y_{p'n'}^n y_{p'n'}^n \right] \]

\[ = \frac{1}{j} \left( -1 \right)^{j-p+j_a+j_o+1} \]

\[ \times \sum_{p,n'} \left[ \left( u_{p'} u_n X_{p'}^{q} - v_{p'} v_n Y_{p'}^{q} \right) u_{p} u_n G(pnp'n' J_{ao}) \right. \]

\[ + \left( v_{p'} v_n X_{p'}^{q} - u_{p'} u_n Y_{p'}^{q} \right) v_{p} u_n F(pnp'n' J_{ao}) \]

\[ + \left( v_{p'} u_n X_{p'}^{q} + u_{p'} v_n Y_{p'}^{q} \right) v_{p} u_n F(pnp'n' J_{ao}) \]

\[ - \left( u_{p'} v_n X_{p'}^{q} + u_{p'} v_n Y_{p'}^{q} \right) v_{p} u_n F(pnp'n' J_{ao}) \]

Above the energies \( E_n \) are the pnQRPA phonon energies, and \( E_n \) and \( E_F \) are the neutron and proton BCS quasiparticle energies. The quantities \( G(pnp'n' J_{ao}) \) and \( F(pnp'n' J_{ao}) \) are the two-body particle–particle and particle–hole interaction matrix elements in the Baranger’s notation [9,10]. The way of construction of the pnMQPM supermatrix from the above listed pieces is the same as for the MQPM and is described in [6].

The general formulation of theory for nuclear beta-decay transitions [13] was compactified and applied to fourth-forbidden non-unique \( \beta^- \) decays in Ref. [1]. Accordingly, the probability of electron emission in the energy interval \( W_e \) to \( W_e + dW_e \) is

\[ P(W_e) dW_e = \frac{G_F^2}{(2\pi)^3} \frac{1}{\hbar^2} C(W_e) p_e C(W_e) (W_0 - W_e)^2 \]

\[ \times F_0(Z, W_e) dW_e, \]

where \( G_F/(\hbar c)^3 \) is the Fermi coupling constant, \( C(W_e) \) is the shape factor, \( W_0 \) is the end-point energy, \( F_0(Z, W_e) \) is the Fermi function and \( p_e \) the momentum of the emitted electron. In this Letter we use the function \( P(W_e) \), scaled by a constant factor, as representative of the spectral shape.

For the numerical work we have used the Coulomb-corrected Woods–Saxon mean-field potential with the parametrization of Ref. [14]. The 1p–0f–1d–0g–0h valence space was adopted for both protons and neutrons. To achieve better results, some of the Woods–Saxon single-particle energies were modified slightly. The modifications were the ones made in Ref. [1]. The two-body interaction matrix elements were obtained from the Bonn one-boson-exchange potential applying \( G \)-matrix techniques [15].

The interaction matrix elements in the BCS calculations were treated as described in Ref. [1]. The pnQRPA spectra were produced by scaling the particle–particle interaction matrix elements separately for each multipole. It turned out that the final log \( f_t \) values were rather insensitive to these pnQRPA adjustments. For the \( ^{113}\text{Cd} \) and \( ^{113}\text{In} \) nuclei we used \( ^{112}\text{Cd} \) as the reference and for the \( ^{115}\text{In} \) and \( ^{115}\text{Sn} \) nuclei the reference was taken to be \( ^{116}\text{Sn} \). Accordingly, the pnQRPA calculations produced phonons in the nuclei \( ^{112}\text{In} \) and \( ^{116}\text{In} \).

Our computed pnMQPM energy spectra are compared with available data in Fig. 1. In the case of the \( A = 115 \) isobars, the final low-energy pnMQPM spectra have roughly the same quality as the MQPM spectra of Ref. [1]. In the case of the \( A = 113 \) isobars, however, an interesting difference was observed: the pnMQPM describes the \( ^{113}\text{Cd} \) in spectrum very nicely, whereas the MQPM spectrum contained many extra low-lying states of mainly three-quasiparticle nature. In the case of \( ^{113}\text{Cd} \) the MQPM spectrum was very close to the experimental spectrum whereas in the pnMQPM the three-quasiparticle states remain somewhat higher. These features indicate that a completely microscopic description of odd-A nuclei in quasiparticle–phonon coupling schemes is a challenging task and the general framework is still open for improvements. However, in this work we are only interested in the wave functions of the lowest states of the four nuclei in Fig. 1. The corresponding computed wave functions are expected to be well described since the theoretical spectra are reasonable below some half of MeV.

The computed half-lives and log \( f_t \) values are presented in Table 1 along with the available experimental data and the results of the previously published MQPM calculations. For both decays the values, calculated using the pnMQPM framework, are very close to the corresponding experimental ones whereas the values calculated using the MQPM approach are of somewhat poorer quality. The reason for this is the following: when compared to the MQPM wave functions, the one-quasiparticle amplitude in the pnMQPM wave function is larger for the \( ^{113}\text{Cd} \) ground state and smaller for the \( ^{113}\text{In} \) ground state. Interestingly, the increase in the ground-state one-quasiparticle amplitude in \( ^{113}\text{Cd} \) overcomes the decrease in that of \( ^{113}\text{In} \), yielding the product of these amplitudes to have a larger absolute value in the pnMQPM. This enhances the contribution of the one-quasiparticle-to-one-quasiparticle transition channel in the pnMQPM calculation, which, in turn, increases the decay rate since the corresponding single-particle transition matrix elements are larger than the ones containing also three-quasiparticle contributions. Similar mechanism works also in the case of the \( A = 115 \) isobars.

The theoretical spectra of the emitted electrons are shown in Fig. 2 for the pnMQPM. In a recent measurement of the \( ^{113}\text{Cd} \) beta decay [18], a severe deviation between the observed spectral shape and the one found in the Table of Isotopes (ToF) web...
Fig. 1. Comparison of the computed pnMQPM energy spectra with the experimental data [16,17].

Table 1

<table>
<thead>
<tr>
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<th>$^{113}\text{Cd}$</th>
<th>$^{115}\text{In}$</th>
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<td>$\log f t$</td>
<td>$T_{1/2}$ (10$^{15}$ yrs)</td>
</tr>
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<td>MQPM</td>
<td>23.45</td>
<td>16.9</td>
</tr>
<tr>
<td>pnMQPM</td>
<td>23.25</td>
<td>10.5</td>
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<tr>
<td>exp. [16,17]</td>
<td>23.20(10)</td>
<td>7.7(3)</td>
</tr>
<tr>
<td>exp. [18]</td>
<td>8.2 ± 0.2(stat.)$^{+0.2}_{-0.1}$(sys.)</td>
<td></td>
</tr>
</tbody>
</table>

page [19] was discovered. The theoretical spectral shape based on the pnMQPM transition matrix elements resembles the experimental one notably more than does the ToI spectrum. There is a distinct bend around 200 keV in both our spectrum and the experimental one, whereas in the ToI spectrum the bend is missing. Because of the relatively high experimental error in the lower energies, a definite conclusion cannot be drawn about whether there indeed exists a valley similar to the one appearing in our predicted theoretical spectrum. The ToI spectrum is, however, dramatically different from the experimental one in that range of energies. More accurate measurements of the electron spectra are to be expected from the COBRA Collaboration [20] in the future.

There exists also a measured beta spectrum for $^{115}\text{In}$ [21] having the maximum value at less than 200 keV, whereas our computed beta spectrum has its maximum at around 300 keV. More accurate measurement of the $^{115}\text{In}$ beta spectrum is called for to see the possible dip-like structure at lower electron energies.

In this Letter the proton–neutron variant of the microscopic quasiparticle–phonon model (pnMQPM) was applied to describe the fourth-forbidden non-unique beta decays of the ground states of $^{113}\text{Cd}$ and $^{115}\text{In}$. The computed half-lives and $\log f t$ values were found to be in excellent agreement with the available experimental data. The shapes of the beta spectra were also calculated and the one for $^{113}\text{Cd}$ was found to be reasonably close to the recent experiment of Ref. [18]. In the case
Fig. 2. Shapes of the theoretical spectra of the emitted electrons for the $\beta^-$ decays of $^{113}$Cd and $^{115}$In. The intensity is given in arbitrary units.

of $^{115}$In full comparison was not possible since the rather old data [21] becomes inaccurate at low electron energies.

Acknowledgements

This work was supported by the Academy of Finland under the Finnish Center of Excellence Program 2006–2011 (Nuclear and Accelerator Based Program at JYFL). M.T. Mustonen is also grateful for the financial support by the Ellen and Artturi Nyysönen Foundation.

References

The smallest known $Q$ value of any nuclear decay: the rare $\beta^-$ decay of $^{115}\text{In} \ (9/2^+) \rightarrow ^{115}\text{Sn} \ (3/2^+)$

by


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Smallest Known $Q$ Value of Any Nuclear Decay: 
The Rare $\beta^-$ Decay of $^{115}\text{In}(9/2^+) \rightarrow ^{115}\text{Sn}(3/2^+)$

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(Received 18 March 2009; published 16 September 2009)

The ground-state-to-ground-state $Q_{\beta^-}$ value of $^{115}\text{In}$ was determined to 497.68(17) keV using a high-precision Penning trap facility at the University of Jyväskylä, Finland. From this, a $Q_{\beta^-}$ value of 0.35(17) keV was obtained for the rare $\beta^-$ decay to the first excited state of $^{115}\text{Sn}$ at 497.334(22) keV. The partial half-life was determined to $4.1(6) \times 10^{20}$ yr using ultra low-background gamma-ray spectrometry in an underground laboratory. Theoretical modeling of this 2nd-forbidden unique $\beta^-$ transition was also undertaken and resulted in $Q_{\beta^-} = 57^{+12}_{-19}$ eV using the measured half-life. The discrepancy between theory and experiment could be attributed to atomic effects enhanced by the low $Q$ value. The present study implies that this transition has the lowest $Q$ value of any known nuclear $\beta$ decay.

From the most recent mass evaluation [1], the energy of the ground-state-to-ground-state $\beta^-$ decay of $^{115}\text{In}$ is 499(4) keV. Given that the energy of the first excited state in $^{115}\text{Sn}$ is 497.334(22) keV [2], the available $\beta^-$ decay energy to the excited state $^{115}\text{Sn}(3/2^+)$ is only 1.7(40) keV. It was thus uncertain whether the $\beta^-$ decay $^{115}\text{In}(9/2^+) \rightarrow ^{115}\text{Sn}(3/2^+)$ was energetically possible. However, Cattadori et al. [3] were first to detect this rare decay in 2005 by measuring the 497.334(22) keV $\gamma$-ray from a 929 g pure indium rod using an ultra low-background $\gamma$-ray spectrometer in the Gran Sasso National Laboratory, 3800 m water equivalent below ground. The measurement was a spin-off from experiments to characterize the bremsstrahlung spectrum of $^{115}\text{In}$, as part of the LENS neutrino project. Figure 1 shows the decay scheme of $^{115}\text{In}$ based on [2] with the inclusion of the $\beta^-$ decay to $^{115}\text{Sn}(3/2^+)$, which is the focus of this study. In the work presented here, this decay is examined through (i) measurements of the cyclotron frequencies of $^{115}\text{In}$ and $^{115}\text{Sn}$ in order to determine the ground-state-to-ground-state $Q_{\beta^-}$ value of $^{115}\text{In}$, which in turn is used to obtain the $Q_{\beta^-}$ value for the rare decay, (ii) underground $\gamma$-ray spectrometry measurements to detect the 497.334 (22) keV $\gamma$-ray from which the partial half-life of the decay is accurately determined, and (iii) a theoretical calculation of the transition energy, the $Q_{\beta^-}$ value, for the rare decay.

The ground-state-to-ground-state $Q_{\beta^-}$ value of $^{115}\text{In}$ to $^{115}\text{Sn}$ was measured with the Penning trap facility [4] at the accelerator laboratory of the University of Jyväskylä. The measurement procedure was similar to $Q$ value measurements of superallowed $\beta$ emitters [5]. The $Q_{\beta^-}$ value was determined by a high-precision comparison of the cyclotron frequencies of the parent and daughter ions:

$$Q_{\beta^-} = m_1 - m_2 = \left(\frac{\nu_2}{\nu_1} - 1\right)(m_2 - q m_e), \quad (1)$$

where $m_1$, $m_2$, and $m_2$ are the masses of the parent atom, the daughter atom, and the electron, $\frac{\nu_2}{\nu_1}$ is the cyclotron frequency ratio of the corresponding ions of charge state $q$, where a $q$ equal to 1+ or 2+ was employed in these studies. The binding energies of the missing atomic electrons can be neglected. The ions were produced using an electric discharge ion source [6]. The composition of the discharging electrode was prepared such that an ion beam of approximately equal intensity was simultaneously produced for both ion species. A purified sample of either $^{115}\text{Sn}$ or $^{115}\text{In}$ ions was prepared using a high-resolution cleaning technique described in [7]. This was done prior to the frequency measurements of the selected species, using the time-of-flight ion-cyclotron resonance technique [8,9] in combination with radio frequency excitation applied with time-separated oscillatory fields [10].

The determination of the cyclotron frequency of one ion species lasted for approximately 1 h and a representative

\[FIG. \ 1. \ \text{Decay schemes of } ^{115}\text{In and } ^{115}\text{Sn}, \text{ showing the decay to the first excited state of } ^{115}\text{Sn}.\]
The cyclotron resonance curve is shown in Fig. 2. The frequency ratio was determined from interleaving measurements of the two species, typically employed to monitor the temporal drift of the magnetic field. In this way, a weighted mean $Q_B$ value could be calculated from several individual frequency ratios. Three sets of data were collected consisting of 11, 15, and 11 individual frequency ratios. Two of the data sets were measured with singly-charged ions while one set was measured with doubly-charged ions. The frequency ratios and $Q$ values obtained are given in Table I. Finally, the $Q_B$ value 0.35(17) keV of the transition $^{115}\text{In}(9/2^+) \rightarrow ^{115}\text{Sn}(3/2^+)$ is deduced from the value in Table I, given that the energy of the excited state $^{115}\text{Sn}(3/2^+)$ is 497.334(22) keV [2]. This indicates that the rare $\beta^-$ decay is energetically possible.

Three $\gamma$-ray spectrometry measurements of a radio-pure indium sample were carried out in the underground laboratory HADES, located at a depth of 500 m water equivalent [11]. The objective was to confirm or dispute the activation of the sample and detectors.

The indium disc was of natural isotopic abundance with 95.71(5)% of the isotope $^{115}\text{In}$ [12]. After surface cleaning the disc dimensions were 10.6 cm (diameter), 4.0 cm (thickness), and 2566.13 g (total mass). Three different ultra low-background $\gamma$-ray spectrometers were made available in HADES for these measurements (measurement time in brackets): Ge-4 (48 days), Ge-7 (15 days) [13], and the sandwich spectrometer (14 days) [13].

The bremsstrahlung spectrum from the ground-state-to-ground-state $\beta^-$ decay is visible in the $\gamma$-ray spectrum in Fig. 3. The excited state of $^{115}\text{In}$ was not present in the sample since the 336 keV $\gamma$-ray line from the main branch was not detected. The peak of interest, 497.334 (22) keV, is located in a favorable energy region with a low background continuum and without interfering background peaks as displayed in Fig. 4. For Ge-4 the count rate of the continuum under the 497 keV peak was 0.38 keV$^{-1}$ d$^{-1}$. Careful analysis confirmed the absence of any unknown $\gamma$-ray emitters in the spectra, which was also the conclusion of Cattadori et al. after a thorough study of alternative sources of the detected 497 keV $\gamma$-ray line. The full energy peak detection efficiencies were calculated using the Monte Carlo technique and the EGS4 simulation software. The geometrical model used for the Monte Carlo simulations was validated by measuring point sources ($^{85}\text{Sr}$, $^{137}\text{Cs}$, $^{134}\text{Cs}$) sandwiched between indium discs of 3, 6, and 12 mm thickness and it was confirmed that the calculated detection efficiencies agree within 3%.

The weighted mean activity of the three measurements was 0.69(9) mBq, which converts to a partial half-life of 4.1(6) $\times$ 10$^{20}$ yr. The relative combined standard uncertainty was determined to 15% and is dominated by counting statistics (14%). Other major contributions to the total uncertainty are detection efficiency (3%), isotopic abundance (0.05%), and internal conversion coefficient (0.12%). The branching ratio was calculated to 1.07(17) $\times$ 10$^{-6}$, which contains an additional uncertainty of 5.7%.

![FIG. 2 (color online). A representative time-of-flight ion-cyclotron resonance obtained for $^{115}\text{Sn}^{2+}$ ions using the excitation time pattern of (50-350-50) ms (On-Off-On).](image)

![FIG. 3 (color online). $\gamma$-ray spectrum of the background (Bkg) and indium on the sandwich spectrometer (SW).](image)

<table>
<thead>
<tr>
<th>Set</th>
<th>No.</th>
<th>Frequency ratio, $Q_B$ (keV)</th>
<th>$Q_B$ value</th>
<th>$Q_B$ ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>11</td>
<td>1.000 004 649 7(27)</td>
<td>497.66(29)</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>15</td>
<td>1.000 004 650 6(23)</td>
<td>497.76(25)</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>11</td>
<td>1.000 004 648 4(37)</td>
<td>497.52(40)</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>12</td>
<td>1.000 004 650 6(23)</td>
<td>497.76(25)</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>13</td>
<td>1.000 004 648 4(37)</td>
<td>497.52(40)</td>
<td></td>
</tr>
</tbody>
</table>
from the half-life of the ground-state-to-ground-state $\beta^-$ decay. The measurements in this work agree with those of Cattadori et al. [3]; see Table II.

The $\beta^-$ decay of the ground state of $^{115}$In to the first excited state in $^{115}$Sn is 2nd-forbidden unique. The half-life of a $K$th forbidden unique $\beta^-$ transition can be written as

$$T_{1/2} = \frac{1}{M^2 f_K(w_0, Z_f, R)}.$$  \hspace{1cm} (2)

Here $M$ is the nuclear matrix element (NME) (equivalent to $M_4$ in Refs. [14,15] containing the nuclear-structure information). $f_K(w_0, Z_f, R)$ is a phase-space function which depends only on the end-point energy $W_0 = w_0 m_e c^2$, the charge $Z_f$ of the daughter nucleus and the nuclear radius $R$. The detailed expression for the function $f_K$ is given in a forthcoming paper.

The NME is calculated using the proton-neutron microscopic quasiparticle-phonon model (pnMQPM) where a realistic microscopic Hamiltonian is diagonalized in a basis consisting of BCS quasiparticles and their couplings with phonons obtained from the proton-neutron quasiparticle random-phase approximation. Further details are given in Ref. [16], where the model was applied to the fourth-forbidden nonunique ground-state-to-ground-state decay of $^{115}$In. The theoretical curve for the half-life of that decay as a function of the $Q$ value is shown in Fig. 5. Five percent random variations in the computed values of the NMEs easily reproduce the measured half-life so that the computed nuclear wave functions of the initial and final states seem very realistic.

Figure 6 shows the end-point energy dependence of the partial half-life for the transition $^{115}$In(9/2$^+$) $\rightarrow$ $^{115}$Sn(3/2$^+$) by using the calculated NME and an estimated 30% uncertainty. Using the measured half-life value, $Q = 57^{+19}_{-12}$ eV is obtained. Here variations of 1 order of magnitude in the value of the NME are needed to reach the 1$\sigma$ interval of measured $Q_{\beta^-}$ value of Fig. 6. Such large

![Figure 4](color online). $\gamma$-ray spectrum of the background (Bkg) and indium on the sandwich spectrometer (SW).

![Figure 5](color online). Calculated half-life of the ground-state-to-ground-state $\beta^-$ decay of $^{115}$In as the function of the end-point energy. The experimental bars combine the half-life range from Ref. [23] and the $Q$ value range from this Letter.

![Figure 6](color online). Calculated relation between the $Q$ value and partial half-life for the $\beta^-$ decay $^{115}$In(9/2$^+$) $\rightarrow$ $^{115}$Sn(3/2$^+$). The grey band indicates a conservative 30% uncertainty estimate for the NME. The experimental bars combine the half-life and $Q$ value ranges extracted in the present underground and trap measurements.

### Table II. Results from the $\gamma$-ray measurements on three spectrometers. The measurement by Cattadori et al. [3] is included for comparison. BR is the branching ratio, $t_{1/2}$ is the partial half-life, and Rel.U is the relative uncertainty.

<table>
<thead>
<tr>
<th>Spectrometer</th>
<th>BR(10$^{-6}$)</th>
<th>$t_{1/2}$(10$^{20}$ yr)</th>
<th>Rel.U(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ge-4</td>
<td>1.06(17)</td>
<td>4.2(7)</td>
<td>17</td>
</tr>
<tr>
<td>Ge-7</td>
<td>1.0(5)</td>
<td>4.4(20)</td>
<td>45</td>
</tr>
<tr>
<td>Sandwich</td>
<td>1.2(4)</td>
<td>3.6(12)</td>
<td>42</td>
</tr>
<tr>
<td>HADES final</td>
<td>1.07(17)</td>
<td>4.1(6)</td>
<td>15</td>
</tr>
<tr>
<td>Cattadori et al.</td>
<td>1.2(3)</td>
<td>3.7(10)</td>
<td>26</td>
</tr>
</tbody>
</table>
variations in the value of the NME are unrealistic and would suggest that the excited state would have an exotic structure. A more plausible reason for the discrepancy is the exceptionally low $Q$ value of the decay.

A small decay energy poses a challenge to the phase-space calculations. One problem is that the electron screening corrections have never been estimated for forbidden beta decays at ultralow $Q$ values. Such estimates exist only for allowed $\beta^-$ decays [17,18], showing an effect of a few percent.

A second source of correction to the phase space is the mismatch between the initial and final atomic states in a $\beta$ decay. In [19], the associated reduction of the decay rate was calculated to be 2% for $Q$ values larger than 20 keV. For $Q$ values of the order of keV the formalism of [19] gives large reductions in decay rates and it breaks down completely for the ultralow $Q$ values such as that discussed in this Letter. This correction thus shifts the shaded theory band in Fig. 6 upwards by an as yet unknown amount. Further corrections come from exchange effects [19,20] and final-state interactions [21]. In [19,20], the exchange effects were computed for $Q$ values larger than 20 keV. The two calculations predict less than 10% effects but in opposite directions so that they contradict each other. Hence, the contribution of the exchange effects is still an open problem even for only moderately small $Q$ values. The molecular final-state interactions were evaluated in [21] for the tritium $\beta$ decay. For heavy nuclei no such evaluations exist and the related effect is a field for continuing work.

The conclusions from the present study are that the combined effects of two very different measurements and the theoretical investigation of the $\beta^-$ transition $^{115}\text{In}(9/2^+) \rightarrow ^{115}\text{Sn}(3/2^+)$ suggest that this rare decay has the lowest $Q$ value discovered for any $\beta$ decay, about 1 order of magnitude smaller than that of $^{187}\text{Re}$ [22]. This Letter also confirms the findings by Cattadori et al. [3]. Additionally, the combination of measurements and theory in this Letter clearly points to the need of further theoretical work on subtle atomic effects entering the area of theoretical modeling of $\beta^-$ transitions with ultralow $Q$ values.

This work was supported by the EU 6th Framework Programme “Integrating Infrastructure Initiative-Transnational Access,” Contract No. 506065 (EURONS, JRA TRAPSPEC) and JRA 2 Contract No. 506222 (ILIAS) as well as the Finnish Center of Excellence Program 2006–2011 (Project No. 213503, Nuclear and Accelerator Based Physics Programme at JYFL).
Forbidden beta decays of $^{96}$Zr and $^{115}$In: Implications for neutrino physics

by

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Forbidden beta decays of $^{96}$Zr and $^{115}$In: Implications for neutrino physics

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Abstract. We summarize our theoretical results for two nuclides of interest for the double-beta decay and neutrino mass studies: $^{96}$Zr and $^{115}$In.

The double-beta decay of $^{96}$Zr competes with three highly-forbidden beta-decay channels. Our microscopic nuclear-structure calculations [1] imply that the half-life of the first-order beta-decay channels is an order of magnitude longer than that of the double-beta decay.

In the work of C. T. Cattadori et al [2] it was discovered that $^{115}$In can beta decay to the first excited state of $^{115}$Sn. It was also suggested that this decay might provide a supplementary way of accessing the neutrino mass. The recent half-life measurement carried out in the underground laboratory HADES confirms the existence and refines the half-life of this decay channel [3]. At the same time the precision mass measurements made at the University of Jyväskylä yield the record-setting ultra-low $Q$ value of 0.35(17) keV [3]. Our theoretical analysis of this decay suggests that atomic effects could play an important role in relating the measured half-life to the measured $Q$ value.

Keywords: forbidden beta decay, beta transition half-lives, low $Q$ value, ultra-low $Q$ value

PACS: 23.40.Hc

THE SINGLE-BETA-DECAY CHANNELS OF $^{96}$Zr

The zirconium isotope $^{96}$Zr is one of the experimentally observed [4] double-beta emitters. The single-beta-decay channels of $^{96}$Zr (see Fig. 1) are energetically allowed, but due to their low $Q$ values and highly forbidden nature have very long partial half-lives and have not yet been experimentally observed. Thus it is of interest to calculate the theoretical partial half-lives of these channels [1].

The involved nuclear wave functions were calculated using the pnQRPA approach (proton-neutron quasiparticle random-phase approximation, see e.g. Ref. [8] for details). The values of the model parameters have been published in Ref. [1]. The general beta-decay formalism is well known (see e.g. Ref. [9] for detailed discussion or Ref. [10] for streamlined ready-to-apply formulas for highly-forbidden non-unique $\beta^-$ decays) for both unique and non-unique beta decays. The unique decays depend solely on one nuclear matrix element, thus allowing simple extraction of log $ft$ values, analogously to the allowed decays. The non-unique decays, instead, depend on several beta nuclear matrix elements and accordingly the spectral shape function is quite complicated.

Our theoretical results for the single-beta-decay channels are summarized on Table 1. These values are different from those published in [1] since the values presented here are recalculated using newer evaluated data [5] for the $Q$ values. The nuclear matrix elements are the ones of Ref. [1] and the conclusions of [1] are confirmed, namely as Table 1 indicates the single-beta decay is dominated by the fourth-forbidden unique
$T^{3}\beta_1/2 > 3.8 \times 10^{19}$ y
$Q_{\beta^-}(g.s.) = 0.161$ MeV
$0^+ \quad 0.000$

$\bullet \quad 96\text{Zr} \quad \bullet \quad 96\text{Nb}$

$T^{\beta\beta}_1/2 = (2.3 \pm 0.2) \times 10^{19}$ y
$0^+ \quad 0.000$

$5^+ \quad 2.975$
$7^+ \quad 2.876$
$6^+ \quad 2.755$
$6^+ \quad 2.441$
$5^+ \quad 2.439$

$\bullet \quad 96\text{Mo}\quad \bullet$

FIGURE 1. The decay scheme of $^{96}\text{Zr}$. The excitation energies (expressed in MeV in the figure), $Q$ values and half-lives are taken from Ref. [5], except for the lower limit of the single-beta-decay half-life and the double-beta-decay half-life, which are from Refs. [6] and [7], respectively.

TABLE 1. Theoretical results for the single-beta-decay channels of $^{96}\text{Zr}$. The $Q$ values are calculated from the experimental energies taken from Ref. [5].

<table>
<thead>
<tr>
<th>Final state</th>
<th>Decay type</th>
<th>$Q$ value (keV)</th>
<th>Partial half-life (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(4^+_1)$</td>
<td>4th-forbidden non-unique</td>
<td>15(5)</td>
<td>$2.3 \times 10^{22}$</td>
</tr>
<tr>
<td>$(5^+_1)$</td>
<td>4th-forbidden unique</td>
<td>117(5)</td>
<td>$2.6 \times 10^{20}$</td>
</tr>
<tr>
<td>$6^+_g\text{s.}$</td>
<td>6th-forbidden non-unique</td>
<td>161(4)</td>
<td>$4.9 \times 10^{29}$</td>
</tr>
</tbody>
</table>

channel to the $5^+_1$ state of $^{96}\text{Nb}$. The partial half-life of this channel, $2.6 \times 10^{20}$ years, is roughly an order of magnitude longer than the experimental double-beta-decay half-life. It is also only slightly less than an order of magnitude longer than the experimental lower limit, $3.8 \times 10^{19}$ years [6].

From our calculation we can draw conclusions concerning the geochemical experiments [11, 12] on the $^{96}\text{Zr}$ half-life. From the computed beta-decay half-life we deduce that the possible contamination from the single-beta-decay channels is still within the experimental uncertainty of the geochemical measurements, and can be so far neglected. The very significant discrepancy between both different geochemical measurements [11, 12] and NEMO experiments [13, 7] for this double-beta-decay half-life still remains an unresolved issue.
THE GROUND-STATE-TO-EXCITED-STATE DECAY OF $^{115}$In

The $^{115}$In decay to the first excited state of $^{115}$Sn, first reported by Cattadori et al in Ref. [2], was confirmed to have the lowest observed beta-decay $Q$ value in the recent JYFLTRAP mass measurement [3]. This $Q$ value, $0.35(17)$ keV, is an order of magnitude lower than that of $^{187}$Re.

As mentioned in the previous section, for the unique beta decays the nuclear information about the initial and final nuclear wave functions is contained in a single nuclear matrix element (NME) $M$. This relation can be written for a $K$th forbidden unique beta decay as

$$T_{1/2} = \frac{1}{M^2 f_K(w_0, Z_f, R)},$$

(1)

where $f_K(w_0, Z_f, R)$ is the phase-space integral depending only on the end-point energy $w_0$, the charge of the daughter nucleus $Z_f$ and the nuclear radius $R$.

In Figure 2 we have plotted our theoretical half-life as a function of the $Q$ value for the $^{115}$In decay and compare it to the experimental data intervals obtained by combining the JYFLTRAP measurement of the $Q$ value with the HADES measurement of the partial half-life. No atomic contributions beyond the usual relativistic Fermi function have been taken into account in the phase space integral. The gray band corresponds to 30% variation in the NME illustrating that the half-life is not very sensitive to uncertainties in the NME and, consequently, uncertainties in the nuclear wave functions.
It can be read from Figure 2 that to shift the theoretical band inside the one-sigma error bars of the experimental data the decay rate should be suppressed by two orders of magnitude. This, in turn, requires a rough order of magnitude suppression in the NME. Since it is very easy to explain both the initial and final state as dominantly one-quasiparticle states even when taking the three-quasiparticle configurations into account by using either the MQPM [10] or pnMQPM [14], it is very unlikely that the NME would be off by an order of magnitude. Because of this fact and the fact that we are dealing with a record-low $Q$ value, it is reasonable to seek the explanation for this discrepancy by taking a closer look at the possible corrections associated to the phase-space integral.

There exists some literature on the various corrections to the decay rate that stem from the atomic electron cloud. The electron screening effect has been traditionally taken into account by modifying the Fermi function according to the Rose prescription [15]. This approach has been tested against self-consistent Hartree-Fock-Slater calculations in Ref. [16] for allowed decays. The self-consistent calculation predicts the screening effect to be roughly 2.5% for allowed $\beta^-$ decays in the limit of zero $Q$ value, while the Rose formula diverges. In the ultra-low-$Q$-value regime the Rose formula gives clearly unphysical values. A more accurate, completely relativistic expression for evaluating the screening effect was derived in Ref. [17]. The $Q$ values as low as this are below that formula’s region of validity as well.

Further contributions may arise from exchange and overlap effects. The imperfect overlap of the initial and final atomic states is expected to hinder the decay. The work of Bahcall [18] suggests that the overlap effect grows stronger as the decay $Q$ value is decreased. The estimated overlap effect for $^{241}\text{Pu}$ ($Q$ value 21 keV) was roughly 2%. In the same article the exchange effect was calculated to have a similar contribution to the decay, hampering the $^{241}\text{Pu}$ decay by an additional 2%.

The work by Harston and Pyper [19] contradicts Bahcall’s results, claiming that the exchange effects might actually enhance the decay. For $^{241}\text{Pu}$ their calculation yields a 7.5% enhancement, hinting that for ultra-low-$Q$-value decays the exchange effect contribution could be very dramatic.

One more possible source for corrections is the contribution from final-state interactions. The molecular final-state has been studied for tritium decay by Saenz and Froelich [20]. It is unclear if their approach would also work for the ultra-low-$Q$-value decays in heavier nuclei.

In conclusion, we have pointed out that the theory on atomic contributions to the nuclear beta decay has so far been developed only for $Q$ values traditionally considered “low”, tens or hundreds of keV and higher. Now that the first observation of an ultra-low-$Q$-value decay has been made, we face the problem that such decays are beyond the reach of approximations designed for the low $Q$ values. Hence new theoretical work is called for. In addition, new experimental work is needed to further improve the $Q$ value measurement of the ultra-low-$Q$-value branch of $^{115}\text{In}$ and to collect data on other possibly existing decays with ultra-low $Q$ values.
ACKNOWLEDGMENTS

This work was supported by the Academy of Finland under the Finnish Center of Excellence Program 2006–2011 (Nuclear and Accelerator Based Program at JYFL).

REFERENCES

7. A. S. Barabash, Average and recommended half-life values for two-neutrino double beta decay: Upgrade ‘09, presentation at MEDEX’09 workshop; see these proceedings (2009).
Nuclear and atomic contributions to beta decays with ultra-low $Q$ values

by

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(Focus Section on Open Problems in Nuclear Structure)

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Nuclear and atomic contributions to beta decays with ultra-low $Q$ values

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Received 25 September 2009
Published 29 March 2010
Online at stacks.iop.org/JPhysG/37/064008

Abstract
Today’s ion trap technology has made it possible to measure ultra-low beta-decay $Q$ values of a few hundred eV and less. Recent measurements of the $^{115}$In beta decay to the first excited state of $^{115}$Sn imply that the theory for such decay needs further developments. The atomic effects that are negligible for most beta decays and introduce minor corrections for the decays with low $Q$ values seem to have dramatic consequences for the beta decays with an ultra-low $Q$ value. We use the ultra-low-$Q$-value decay branch of $^{115}$In to illustrate the problem and point out areas where new theory needs to be developed.

1. Introduction
Traditionally, beta-decay $Q$ values in the energy range of tens to hundreds of keV have been considered low. The methods in experimental nuclear physics have progressed to the point where it is now possible to observe decays with a $Q$ value as low as few hundreds of eV.

In this paper we introduce the case of the record-setting $Q$ value of the beta-decay branch of $^{115}$In (figure 1) and use it to illustrate that in this energy range the atomic contributions are no longer minor corrections. We take a brief look at the past research on these effects and point out that experiments have moved past the tested region of validity of old estimates to the regime where some of the formulae derived in the past for decays with low $Q$ values produce corrections that are clearly unphysical for the presently measured $Q$ values. Since it appears that the theoretical description of decays with hundreds of eV requires using better approximations than the decays that have traditionally been called ‘low’, we have chosen to call such $Q$ values ultra-low.

2. The ultra-low-$Q$-value decay of $^{115}$In
The beta decay of the $^{115}$In ground state to the first excited state of $^{115}$Sn was first discovered by Cattadori et al [1]. The existence of this tiny decay channel with the branching ratio of
only 1.07(17) \times 10^{-6} was recently confirmed by a combined effort of HADES underground laboratory and JYFLTRAP Penning trap measurements [2], and independently by Mount et al [3]. The JYFLTRAP mass measurement revealed that this decay channel has a beta-decay $Q$ value of 0.35(17) keV and the experiment [3] further refined the value down to 0.155(24) keV. It is now clear that the $Q$ value of this decay is roughly an order of magnitude smaller than the lowest previously known $Q$ value of $^{187}$Re ($Q_{\beta^-} = 2.469(4)$ keV [4]).

From the theoretical point of view, this decay to the excited state is a second-forbidden unique beta decay. In practice, this means that, unlike in the case of non-unique forbidden decays, the dependence of the half-life on the nuclear structure is extraordinarily simple. It is contained in a single nuclear matrix element $M$. This relation can be written for a $K$th forbidden unique beta decay as

$$ T_{1/2} = \frac{1}{M^2 f_K(w_0, Z_f, R)}, $$

where $f_K(w_0, Z_f, R)$ is the phase-space integral depending only on the electron end-point energy $w_0$, the final nuclear charge $Z_f$ and the nuclear radius $R$ [5].

Figure 2 illustrates the disagreement between our theoretical calculation and the newest experimental results. The grey band corresponds to an ad hoc 30% variation in the nuclear matrix element. From figure 2, we note that to reach the 1σ limits of the experiments, we need at least a factor of 15 longer theoretical half-life. In order to associate this discrepancy with solely an inaccurate nuclear matrix element, the matrix element would have to be off by a factor of 4.

Both the initial and final state of this decay can be easily explained as a one-particle and a one-hole state in a simple shell-model approach, or as one-quasiparticle states in the nuclear BCS. When extending the configuration space to the three-quasiparticle components by using the MQPM (microscopic quasiparticle-phonon model [6]) or the pnMQPM (proton–neutron MQPM [7]) as the nuclear model, these states still remain almost pure one-quasiparticle states. The fourth-forbidden non-unique ground-state-to-ground-state decay between these very same nuclei was successfully described using the MQPM in [8] and yet slightly better using the pnMQPM in [7]. This still does not exclude the possibility that the 3/2$^+$ state in $^{115}$Sn could have a more exotic structure, e.g. by opening the proton shell by two-particle-two-hole excitations, a typical way of producing intruding deformed low-energy nuclear states. Only
Figure 2. Disagreement between the latest experimental measurements [2, 3] and our theoretical calculation. The data point labelled JYFLTRAP is the joint measurement of [2] and the one labelled BJM is the $Q$-value from [3]. Both of them have been combined with the HADES half-life measurement [2]. The overlap of the $1\sigma$ $Q$-value error bars is 4 eV. The grey band corresponds to a 30% variation in the beta-decay nuclear matrix element. No atomic electron effects have been taken into account in the theoretical calculation.

further calculations, e.g. by the nuclear shell model, can tell if our interpretation is correct or not. Hence, at the moment it is not certain what portion of the factor of 4 difference emerges from the nuclear-structure corrections or the atomic corrections. This uncertainty and the fact that the $Q$ value is notably lower than in the cases of the so-called low-$Q$-value decays motivates us to turn our attention to effects that have been neglected thus far for decays with extremely small $Q$ values. The presently discussed ultra-low $Q$ values could introduce dramatic corrections to the thus far used theoretical approximations.

3. Previous work on atomic effects

The classic recipe for taking the electron screening corrections into account is to modify the Fermi function in the phase-space integral following the Rose formula [9]. The validity of the Rose formula for allowed $\beta$ decays has been tested in [10] by comparing the corrected Fermi function to the Fermi function extracted from self-consistent Hartree–Fock–Slater calculations. It turns out that in the limit of zero $Q$ value, the self-consistent correction for allowed $\beta^-$ decay is roughly 2.5%, but the Rose prescription diverges. It is clear that the Rose formula is unreliable for ultra-low $Q$ values.

A completely relativistic expression for the electron screening corrections (basically a refinement of the Rose formula) was derived in [11]. It was applied to superallowed $\beta$ emitters with $Q$ values higher than 2 MeV, where the corrections were of the order of 0.2%. The refinements do not make this formula applicable to the ultra-low-$Q$-value regime. In fact, if one blindly applies the formula for the ultra-low-$Q$-value decay of $^{115}$In, one ends up with a huge negative half-life.

Bahcall has derived formulae for approximating the atomic overlap and exchange effects [12] in both $\beta^\pm$ decays and electron capture. His results for the corrections in the case of
low-$Q$-value $\beta^-$ decays suggested that the corrections would be larger for smaller $Q$ values. The lowest $Q$ value he applied his method to was 21 keV for $^{241}$Pu beta decay. There the magnitude of both the exchange and overlap contributions was about 2%, totaling 4%. The approximations used by Bahcall do not work in the ultra-low regime below a few hundred eV.

Further work on exchange effects by Harston and Pyper [13] contradicts Bahcall’s results. According to their work, exchange effects might actually enhance the decay. Their approach resulted in a 7.5% enhancement of the decay rate for $^{241}$Pu due to the exchange effects.

One more possibly significant effect rises from final-state interactions. Molecular final-state interactions have been carefully studied for the decay of tritium by Saenz and Froelich in [14]. It is not obvious if the validity of their approach extends also to the regime of ultra-low $Q$ values. Clarification of this remains yet another open problem.

4. Conclusions

The existing methods to deal with the different atomic corrections to the nuclear beta-decay rates have been developed in past decades to estimate the magnitudes of the corrections in the cases of what have traditionally been called as ‘low’ $Q$ values. Progress in the experimental techniques has made it possible to measure $Q$ values that are orders of magnitude smaller. The recent discovery of the ultra-low-$Q$-value branch of $^{115}$In and the related theoretical calculations suggests that the effects stemming from atomic electrons have not only non-negligible but possibly dramatic consequences for the (partial) half-lives. The past research on such effects has been dealing with ‘only’ low-$Q$-value beta emitters, and the related methods are seen to break down in the ultra-low-$Q$-value regime.

The atomic effects have been shown to grow stronger as the $Q$ value drops on the low-$Q$-value regime, but to determine if this trend continues in the ultra-low regime new theoretical work is needed. We cannot rule out the possibility that the $^{115}$In first excited state has an exotic configuration, but even if it has, the fact that these effects have not yet been sufficiently investigated remains. Now that the experimentalists have succeeded in pushing into the ultra-low-$Q$-value regime, the theory side should also be extended to cover it.

From the theoretical point of view, the study of beta emitters with ultra-low $Q$ values provides an interesting challenge at the borderline between the atomic and nuclear physics. For experimentalists it remains a challenge to observe more decays with an ultra-low $Q$ value to allow testing of the future developments of the theory against a reasonable amount of experimental data.

Acknowledgment

This work was supported by the Academy of Finland under the Finnish Center of Excellence Program 2006–11 (Nuclear and Accelerator Based Program at JYFL).

References